Non-deterministic and Stochastic Selection Functions

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Agenda

- Spector's bar recursion
  (functional interpretation of $\mathbf{AC}_0$)
- A game-theoretic interpretation
  (selection functions)
- Non-deterministic players
  (the powerset monad)
- Stochastic players
  (the probability monad)

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Interpreting $\mathbf{AC}_0$ (classically)

The Principles.

$$\begin{aligned} \mathbf{LEM} &: \forall n^{\mathbb{N}}(A_n \vee \neg A_n) \\ \mathbf{CA} &: \exists f^{\mathbb{N}\to\mathbb{B}} \forall n^{\mathbb{N}}(f(n) \leftrightarrow A_n) \\ \mathbf{AC}_0 &: \forall n^{\mathbb{N}} \exists x^{\sigma} A_n(x) \to \exists \alpha^{\sigma^{\mathbb{N}}} \forall n\, A_n(\alpha_n) \\ \mathbf{DNS} &: \forall n^{\mathbb{N}} \neg\neg A_n \to \neg\neg \forall n^{\mathbb{N}} A_n \end{aligned}$$

Spector'62.

- $\mathbf{LEM} + \mathbf{AC}_0 \vdash \mathbf{CA}$
- If $\mathbf{LEM} \vdash A$ then $\vdash A^N$ (negative translation)
- $\mathbf{DNE} + \neg\neg\mathbf{AC}_0 \vdash (\mathbf{AC}_0)^N$
- Bar recursion solves Dialectica interpretation of **DNE**

Interpreting $\mathbf{AC}_0$ (classically)

Challenge.

Functional (Dialectica) interpretation of

$$\forall n^{\mathbb{N}} \neg\neg \exists x^{\sigma} \forall y^{\tau} A_n(x, y) \to \neg\neg \exists \alpha^{\sigma^{\mathbb{N}}} \forall n, y\, A_n(\alpha_n, y)$$

Problem A (solves challenge).

Given $\phi_n^{(\sigma \to \tau) \to \sigma}$, $\omega^{\sigma^{\mathbb{N}} \to \mathbb{N}}$ and $q^{\sigma^{\mathbb{N}} \to \tau}$ find $\alpha^{\sigma^{\mathbb{N}}}$, $n^{\mathbb{N}}$ and $p^{\sigma \to \tau}$ such that

$$A_n(\phi_n(p), p(\phi_n(p))) \;\to\; A_m(\alpha_m, q(\alpha))$$

where $m = \omega(\alpha)$.

Interpreting $\mathbf{AC}_0$ (classically)

Problem A (solves challenge).

Given $\phi_n^{(\sigma\to\tau)\to\sigma}$, $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau}$ find $\alpha^{\sigma^{\mathbb{N}}}$, $n^{\mathbb{N}}$ and $p^{\sigma\to\tau}$ such that

$$A_n(\phi_n(p), p(\phi_n(p))) \to A_m(\alpha_m, q(\alpha))$$

where $m = \omega(\alpha)$.

Problem B (solution for B gives solution for A).

Given $\phi_n^{(\sigma\to\tau)\to\sigma}$, $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau}$ find sequences $\alpha^{\sigma^{\mathbb{N}}}$ and $p_n^{\sigma\to\tau}$ such that

$$\alpha_n = \phi_n(p_n)$$
$$p_n(\alpha_n) = q(\alpha)$$

for $n \leq \omega(\alpha)$.

Interpreting $\mathbf{AC}_0$ (classically)

Problem B (solution for B gives solution for A).

Given $\phi_n^{(\sigma\to\tau)\to\sigma}$, $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau}$ find sequences $\alpha^{\sigma^{\mathbb{N}}}$ and $p_n^{\sigma\to\tau}$ such that

$$\alpha_n = \phi_n(p_n)$$
$$p_n(\alpha_n) = q(\alpha)$$

for $n \leq \omega(\alpha)$.

Solution for B (bar recursion).

Define $B(s)\colon\ \sigma^{\mathbb{N}}$ as follows $(s\colon\ \sigma^*$ and $\hat{s} = s * 0\ldots)$

$$B(s) = \begin{cases} \hat{s} & \text{if } \omega(\hat{s}) < |s| \\ B(s * c) & \text{if } \omega(\hat{s}) \geq |s| \end{cases}$$

where $c =_\sigma \phi_{|s|}(\kappa_s)$ and $\kappa_s(x) =_\tau q(B(s * x))$.

Take $\alpha = B(\langle\ \rangle)$ and $p_n = \kappa_{\overline{\alpha}(n)}$, where $\overline{\alpha}(n) = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1}\rangle$.

Interpreting $\mathbf{AC}_0$ (classically)

Solution for B (bar recursion).

Define $B(s) \colon \sigma^{\mathbb{N}}$ as follows ($s \colon \sigma^*$ and $\hat{s} = s * 0\ldots$)

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Main Lemmas (Spector'62).

- If $\alpha = B(\langle\ \rangle)$ then $\alpha = B(\overline{\alpha}(n))$, for all n (hence $p_n(\alpha_n) = q(\alpha)$)
- If $\alpha = B(\langle\ \rangle)$ and $n = \omega(\alpha)$ then $\omega(\overline{\alpha, n}) \geq n$
- If $\alpha = B(\langle\ \rangle)$ and $n = \omega(\alpha)$ then $\alpha_n = \phi_n(\kappa_{\overline{\alpha}(n)})$ (hence $\alpha_n = \phi_n(p_n)$)

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Game-theoretic interpretation

Problem C.

Let formula $A_n(x^\sigma, y^\tau)$ be given. Suppose $\phi_n^{(\sigma\to\tau)\to\sigma}$ are such that, for all $n^{\mathbb{N}}$ and $p^{\sigma\to\tau}$,

$$A_n(\phi_n(p), p(\phi_n(p))).$$

Then, given $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau}$ find $\alpha^{\sigma^{\mathbb{N}}}$ such that, for all $n \leq \omega(\alpha)$,

$$A_n(\alpha_n, q(\alpha)).$$

Game-theoretic interpretation.

Think of x^σ as a <u>**move**</u> at round n, and of y^τ as the <u>**outcome**</u> of a game

Think of $A_n(x^\sigma, y^\tau)$ as saying that x is a <u>**good move**</u> relative to the outcome y

Think of $p^{\sigma\to\tau}$ as a <u>**game continuation**</u> (predict outcome of any given move)

Premise: $\phi_n^{(\sigma\to\tau)\to\sigma}$ finds a <u>**good move**</u> given a <u>**game continuation**</u> (selection function)

Conclusion: $\alpha^{\sigma^{\mathbb{N}}}$ is a <u>**good play**</u> up to $\omega(\alpha)$

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$$A_n(\phi_n(p), p(\phi_n(p))).$$

Then, given $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau}$ find $\alpha^{\sigma^{\mathbb{N}}}$ such that, for all $n \leq \omega(\alpha)$,

$$A_n(\alpha_n, q(\alpha)).$$

Solution for C (bar recursion).

Same $B(s)\colon\ \sigma^{\mathbb{N}}$ as before does the job.

We can think of $B(s)$ as an iteration of a binary operation (on selection functions)

$$\otimes\colon J_\tau\sigma \times J_\tau\sigma \to J_\tau(\sigma\times\sigma)$$

where $J_\tau\sigma$ abbreviates the type $(\sigma\to\tau)\to\sigma$.

Game-theoretic interpretation

Solution for C (bar recursion).

Same $B(s)\colon \sigma^{\mathbb{N}}$ as before does the job.

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$$\otimes\colon J_\tau\sigma \times J_\tau\sigma \to J_\tau(\sigma \times \sigma)$$

where $J_\tau\sigma$ abbreviates the type $(\sigma \to \tau) \to \sigma$.

Product of selection functions (Escardó/O.'2010).

Given $\phi, \psi\colon (\sigma \to \tau) \to \sigma$ and $q\colon \sigma \times \sigma \to \tau$ let

$$\kappa(x^\sigma) =_\sigma \psi(\lambda y \,.\, q(x, y))$$

and

$$a =_\sigma \phi(\lambda x \,.\, q(x, \kappa(x))$$

so that $(\phi \otimes \psi)(q) =_{\sigma\times\sigma} (a, \kappa(a))$.

Higher-order games

Definition (Escardó/O.'2010).

A (sequential) **higher-order game** is defined by:

- a type X_n for each round $n : \mathbb{N}$
- an outcome type R
- an outcome function $q : \Pi_n X_n \to R$
- selection functions $\phi_n : (X_n \to R) \to X_n$, for each round $n : \mathbb{N}$
- a termination function $\omega : \Pi_n X_n \to \mathbb{N}$

Remarks.

- $\phi_n(p) : X_n$ is the "best move" in game continuation $p : X_n \to R$
- for each play $\alpha : \Pi_n X_n$, only the initial segment up to $\omega(\alpha)$ is relevant
- if termination is not assumed R needs to be discrete and q continuous

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Selection functions and monads

Theorem (Escardó/O.'2017).

For any strong monad M let $J^M_\tau \sigma$ abbreviate the type $(\sigma \to M\tau) \to M\sigma$.

$J^M_\tau \sigma$ is also a strong monad. In particular, we also have a binary operation:

$$\otimes \colon J^M_\tau \sigma \times J^M_\tau \sigma \to J^M_\tau (\sigma \times \sigma).$$

Product of selection functions (Escardó/O.'2017).

Given $\phi, \psi \colon (\sigma \to M\tau) \to M\sigma$ and $q \colon \sigma \times \sigma \to M\tau$ let (where $q^* \colon \sigma \times M\sigma \to M\tau$)

$$\kappa(x^\sigma) =_{M\sigma} \psi(\lambda y^\sigma \,.\, q(x, y))$$

and

$$a =_{M\sigma} \phi(\lambda x \,.\, q^*(x, \kappa(x))$$

so that $(\phi \otimes \psi)(q) =_{M(\sigma\times\sigma)} a \otimes_M \kappa^*(a)$.

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Game-theoretic interpretation.

Consider the finite powerset monad $(M(\tau) = \tau^*)$

Selection functions then are "multi-valued"

Before: $\phi_n^{(\sigma \to \tau) \to \sigma}$ finds a **good move** given a **game continuation** (selection function)

Now: $\phi_n^{(\sigma \to \tau^*) \to \sigma^*}$ finds a **set of moves** containing a good move

Non-deterministic players

Problem D.

Let formula $A_n(x^\sigma, y^{\tau*})$ be given, anti-monotone in $y^{\tau*}$.

Suppose $\phi_n^{(\sigma\to\tau*)\to\sigma*}$ are such that, for all $n^{\mathbb{N}}$ and $p^{\sigma\to\tau*}$,

$$\exists x \in \phi_n(p)\, A_n(x, p(x)).$$

Then, given $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau*}$ find finite set $A \subset \sigma^{\mathbb{N}}$ such that

$$\exists \alpha \in A\, \forall n \leq \omega(\alpha)\, A_n(\alpha_n, q(\alpha)).$$

Game-theoretic interpretation.

Premise: $\phi_n^{(\sigma\to\tau*)\to\sigma*}$ finds a finite set of moves containing a **good move** for round n

Conclusion: A is a finite set which contains **good play**

Non-deterministic players

Problem D.

Let formula $A_n(x^\sigma, y^{\tau^*})$ be given, anti-monotone in y^{τ^*}.

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$$\exists x \in \phi_n(p)\, A_n(x, p(x)).$$

Then, given $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau^*}$ find finite set $A \subset \sigma^{\mathbb{N}}$ such that

$$\exists \alpha \in A\, \forall n \leq \omega(\alpha)\, A_n(\alpha_n, q(\alpha)).$$

Solution for D (bar recursion with finite sets).

Define $B(s) \subset \sigma^{\mathbb{N}}$ as follows ($s\colon\ \sigma^*$ and $\hat{s} = s * 0\ldots$)

$$B(s) = \begin{cases} \{\hat{s}\} & \text{if } \omega(\hat{s}) < |s| \\ \cup_{c\in C}\, B(s * c) & \text{if } \omega(\hat{s}) \geq |s| \end{cases}$$

where $C =_{\sigma^*} \phi_{|s|}(\kappa_s)$ and $\kappa_s(x) =_{\tau^*} \cup_{\alpha\in B(s*x)}\, q(\alpha)$. Take $A = B(\langle\ \rangle)$.

Herbrand interpretation of $\mathbf{AC}_0$

Solution for D (bar recursion with finite sets).

Define $B(s) \subset \sigma^{\mathbb{N}}$ as follows ($s\colon \sigma^*$ and $\hat{s} = s * 0...$)

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where $C =_{\sigma^*} \phi_{|s|}(\kappa_s)$ and $\kappa_s(x) =_{\tau^*} \cup_{\alpha \in B(s*x)} q(\alpha)$. Take $A = B(\langle\, \rangle)$.

Theorem (Escardó/O.'2017).

The above construction solves the Herbrand functional interpretation of **DNS** and hence $\mathbf{AC}_0$ (classically).

Egorov Theorem

Bounded monotone sequences.

For monotone sequences $(a_n)_{n \in \mathbb{N}} \in [0,1]$ and $\delta > 0$ and $\phi_\delta(p) = \{p^k(0) \mid k \leq \lceil 1/\delta \rceil\}$

$$\forall p^{\mathbb{N} \to \mathbb{N}} \exists m^{\mathbb{N}} \in \phi_\delta(p)\ (x_m - x_{p(m)} \leq \delta).$$

Theorem (Avigad et al'2011).

$(X_n)_{n \in \mathbb{N}}$ random variables. Assume $\omega^{\mathbb{N}^{\mathbb{N}} \to \mathbb{N}}$ is such that, for all $\alpha^{\mathbb{N}^{\mathbb{N}}}$,

$$\mathbb{P}\left(\exists n \leq \omega(\alpha) \forall i, j \in [n, \alpha_n] \left(X_i - X_j \leq \varepsilon \right) \right) = 1.$$

Given $h^{\mathbb{N} \to \mathbb{N}}$ and $\delta > 0$ let $q(\alpha) = h(\omega(\alpha))$, and $B(\langle\,\rangle) \subset \mathbb{N}^{\mathbb{N}}$ as before. Then $N = \max_{\alpha \in B(\langle\,\rangle)} \omega(\alpha)$ is such that for some $n \leq N$

$$\mathbb{P}\left(\forall i, j \in [n, h(n)] \left(X_i - X_j \leq \varepsilon \right) \right) > 1 - \delta.$$

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Bar recursion with finite sets (and max : $\mathbb{N}^* \to \mathbb{N}$ algebra).

Define $B(s) \subset \mathbb{N}^{\mathbb{N}}$ as follows

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where $C =_{\mathbb{N}^*} \phi_{|s|}(\kappa_s)$ and $\kappa_s(x) =_{\mathbb{N}} \max\{q(\alpha) \mid \alpha \in B(s * x)\}$.

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Game-theoretic interpretation.

Consider the probability (Giry) monad ($G(\tau)$ = probability measures on τ)

Selection functions pick moves at random

Before: $\phi_n^{(\sigma \to \tau) \to \sigma}$ finds a <u>**good move**</u> given a <u>**game continuation**</u> (selection function)

Now: $\phi_n^{(\sigma \to G\tau) \to G\sigma}$ choose a <u>**probability measure over moves**</u>

Playing (optimally) against irrational players

Problem E.

Think of $\phi_n^{(\sigma \to G\tau) \to G\sigma}$ as stochastic players.

Given outcome function $q^{\sigma^* \to \tau}$ of the game, find best opening move (maximising chance of winning).

Solution for E (bar recursion with probability measures).

Map $\sigma \mapsto G(\sigma)$ is the **Giry monad** (Lawvere'62) given by

$$\eta \colon \sigma \to G(\sigma)$$
$$\eta(x) = \mathrm{Dirac}(x)$$

and ($\mu(Q)(U) \simeq$ average/expected measure of measurable subset $U \subseteq \sigma$)

$$\mu \colon G(G(\sigma)) \to G(\sigma)$$
$$\mu(Q)(U) = \int_{\mathbb{P} \in G(\sigma)} \mathbb{P}[U] \, dQ(\mathbb{P})$$

Work in progress...

- Haskell implementation of calculation of optimal moves in stochastic games (finite X_n and R)
- Assuming $\phi_n \colon (X_n \to R^*) \to X_n^*$ (non-empty finite sets monad) calculates all good moves, have shown (Agda implementation) that monadic product calculates all optimal plays (see Martín's TypeTopology git https://github.com/martinescardo/TypeTopology)
- Haskell implementation of $\alpha\beta$-pruning search optimisation using reader monad $TX = R \to X$

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