On the Proof Theory of the Selection Monad

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Mathematics

Analysis
Comprehension / Choice

Arithmetic
Induction

Foundation
Logic

Mathematics ⇒ Computation

Analysis
Comprehension / Choice

Arithmetic
Induction

Foundation
Logic

Computation

$$\exists f^{\mathbb{N}\to\mathbb{N}} \forall n (f(n) = 0 \leftrightarrow A(n)) \quad \text{(CA)}$$

$$A(0) \wedge \forall n^{\mathbb{N}}(A(n) \to A(n+1)) \to \forall n^{\mathbb{N}} A(n) \quad \text{(IND)}$$

$$A \vee \neg A \quad \text{(LEM)}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge \mathrm{I}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ (\to \mathrm{I}) \qquad \frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \vee A_2}\ (\vee \mathrm{I})$$

$$\frac{\Gamma \vdash A_1 \wedge A_2}{\Gamma \vdash A_i}\ (\wedge \mathrm{E}) \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}\ (\to \mathrm{E}) \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}\ (\vee \mathrm{E})$$

$$\frac{}{\Gamma, A \vdash A}\ (\mathrm{Ax}) \qquad \frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x A(x)}\ (\forall \mathrm{I}) \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ (\exists \mathrm{I})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\mathrm{EFQ}) \qquad \frac{\Gamma \vdash \forall x A(x)}{\Gamma \vdash A(t)}\ (\forall \mathrm{E}) \qquad \frac{\Gamma \vdash \exists x A(x) \quad \Gamma, A(x) \vdash B}{\Gamma \vdash B}\ (\exists \mathrm{E})$$

Agenda

- Negative translations

  Taming negation...

- Hilbert's ε-calculus and Gödel's Dialectica

  Taming quantifiers...

- Arithmetic

  Interpreting induction

- Analysis

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Excluded Middle "Tamed"

$$\boxed{\vdash A \vee \neg A}$$

$$\boxed{\vdash \neg\neg(A \vee \neg A)}$$

$$\dfrac{\dfrac{[A]_\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\dfrac{\dfrac{\dfrac{\bot}{\neg A}\,\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\dfrac{\bot}{A \vee \neg A}\,\text{PBC}, \gamma}}$$

$$\dfrac{\dfrac{[A]_\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\dfrac{\dfrac{\dfrac{\bot}{\neg A}\,\alpha}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_\gamma}{\dfrac{\bot}{\neg\neg(A \vee \neg A)}\,{\rightarrow}\text{I}, \gamma}}$$

Kolmogorov Translation

Theorem (Kolmogorov 1925).

If **CL** proves $\Gamma \vdash A$ then **IL** proves $\Gamma^K \vdash A^K$

$$
\begin{aligned}
(A \wedge B)^K &\equiv \neg\neg(A^K \wedge B^K) & \quad (P(\vec{x}))^K &\equiv \neg\neg P(\vec{x}) \\
(A \vee B)^K &\equiv \neg\neg(A^K \vee B^K) & \quad (\forall x A)^K &\equiv \neg\neg \forall x A^K \\
(A \rightarrow B)^K &\equiv \neg\neg(A^K \rightarrow B^K) & \quad (\exists x A)^K &\equiv \neg\neg \exists x A^K
\end{aligned}
$$

Crucial.

IL proves:

$$A \vdash \neg\neg A \qquad \eta_A : A \to TA$$

$$\neg\neg A,\ A \to \neg\neg B \vdash \neg\neg B \qquad \beta_{A,B} : TA \to (A \to TB) \to TB$$

$$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \qquad \otimes_{A,B} : TA \times TB \to T(A \times B)$$

Continuation and Selection Monads

$$K_R X = (X \to R) \to R$$

continuation monad

$$\eta^K_X \colon X \to K_R X$$

$$\beta^K_{X,Y} \colon K_R X \to (X \to K_R Y) \to K_R Y$$

$$\otimes^K_{X,Y} \colon K_R X \times K_R Y \to K_R (X \times Y)$$

$$J_R X = (X \to R) \to X$$

selection monad

$$\eta^J_X \colon X \to J_R X$$

$$\beta^J_{X,Y} \colon J_R X \to (X \to J_R Y) \to J_R Y$$

$$\otimes^J_{X,Y} \colon J_R X \times J_R Y \to J_R (X \times Y)$$

Negative Translations

M. Escardó and P. Oliva
The Peirce translation
Annals of Pure and Applied Logic, 163(6):681-692, 2012
Translations eliminating Peirce's law $((A \to B) \to A) \to A$ *(minimal logic)*

G. Ferreira and P. Oliva
On the relation between various negative translations
Ontos-Verlag Mathematical Logic Series, vol 3, 227-258, 2012
Gödel-Gentzen and Kuroda as "simplifications" of Kolmogorov

R. Arthan and P. Oliva
Double negation semantics for generalisations of Heyting algebras
Studia Logica, vol 109, pages 341–365, 2021
Negative translations in absence of full contraction (sub-structural logic)

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Hilbert's ε-calculus

Idea (Hilbert, 1921).

For each formula $A(x)$ with a distinct free-variable x add a new term $\varepsilon_x A(x)$ with axioms:

$$(\dagger) \qquad A(x) \rightarrow A(\varepsilon_x A(x))$$

Any formula is equivalent to a quantifier-free formula. Let $\mathbf{CL}_\varepsilon$ denote ε-calculus.

Theorem (ε-Theorem).

Suppose Γ, A do not involve ε-terms. If $\Gamma \vdash_{\mathbf{CL}_\varepsilon} A$ then $\Gamma \vdash_{\mathbf{CL}} A$.

The ε-substitution method (arithmetic, intuition).

- Take $(\dagger)$ $A(n) \rightarrow A(\varepsilon_n A(n)) \wedge \varepsilon_n A(n) \leq n$ *(least witness)*
- Start with $\varepsilon_n A(n) = 0$
- If $(\dagger)$ fails for $n = t$ then $A(t)$ but $\neg A(\varepsilon_n A(n))$ or $\varepsilon_n A(n) > t$, take $\varepsilon_x A(x) = t$, iterate

Hilbert's ε-calculus

$$\exists_x : (\mathbb{N} \to \mathbb{B}) \to \mathbb{B}$$

continuation monad

$$A(x) \quad \mapsto \quad \exists x A(x)$$

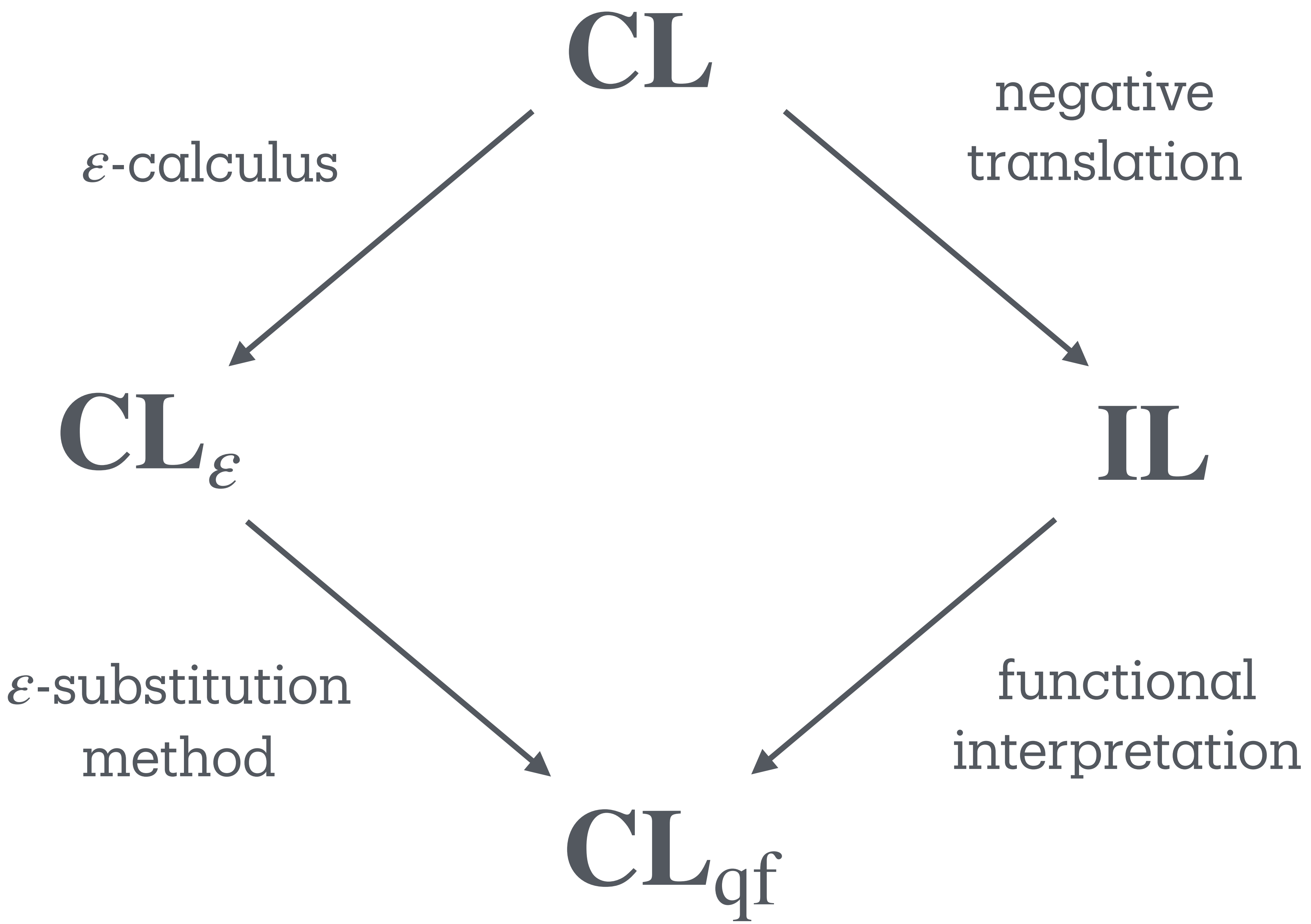
$$\varepsilon_x : (\mathbb{N} \to \mathbb{B}) \to \mathbb{N}$$

selection monad

$$A(x) \quad \mapsto \quad \varepsilon_x A(x)$$

$$\exists x A(x) \Leftrightarrow A(\varepsilon_x A(x))$$

Hilbert vs Gödel



Gödel's Functional (Dialectica) Interpretation

$$A \;\mapsto\; \exists x^X \forall y^R\, A_D(x;y) \;\mapsto\; A_D(t;y)$$

$$\neg\neg\exists x^X \forall y^R A_D(x;y)$$

$$\mapsto \quad \neg\forall x^X \exists y^R \neg A_D(x;y)$$

$$\mapsto \quad \neg\exists p^{X\to R} \forall x^X \neg A_D(x;p(x))$$

$$\mapsto \quad \forall p^{X\to R} \exists x^X \neg\neg A_D(x;p(x))$$

$$\mapsto \quad \exists \varepsilon^{(X\to R)\to X} \forall p^{X\to R} \neg\neg A_D(\varepsilon(p);p(\varepsilon(p)))$$

selection function

Gödel's Functional (Dialectica) Interpretation

$$A \mapsto \exists x^{X} \forall y^{R} A_D(x;y) \mapsto A_D(t;y)$$

$$A^{+} \qquad A^{-} \qquad |A|^{x}_{y}$$

Gödel's Functional (Dialectica) Interpretation

Definition.

Functional interpretations associate formulas A to triples $(A^+, A^-, |A| \subseteq A^+ \times A^-)$

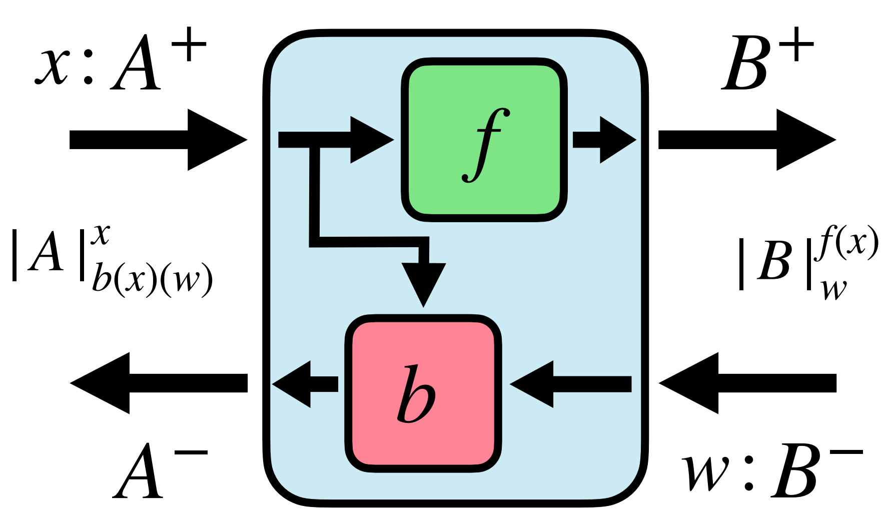
- A^+ is the set/type of evidence for A
- A^- is the set/type of counter-evidence for A
- $|A|^x_y$ determines whether evidence $x \in A^+$ wins against counter-evidence $y \in A^-$

Definition (Interpreting implication).

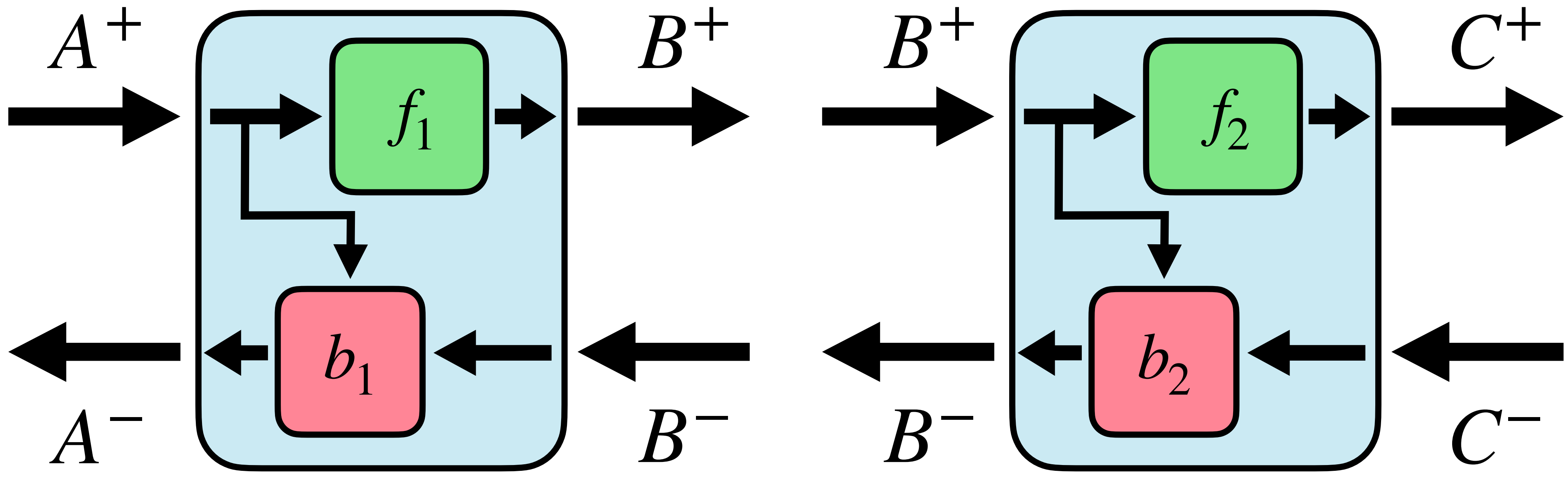
Implication $A \to B$ is interpreted as a pair of maps:

- $f\colon A^+ \to B^+$ (evidence map)
- $b\colon A^+ \to B^- \to A^-$ (counter-evidence map)
- for $x \in A^+$ and $w \in B^-$, we must have

$$|A|^x_{b(x)(w)} \to |B|^{f(x)}_w$$

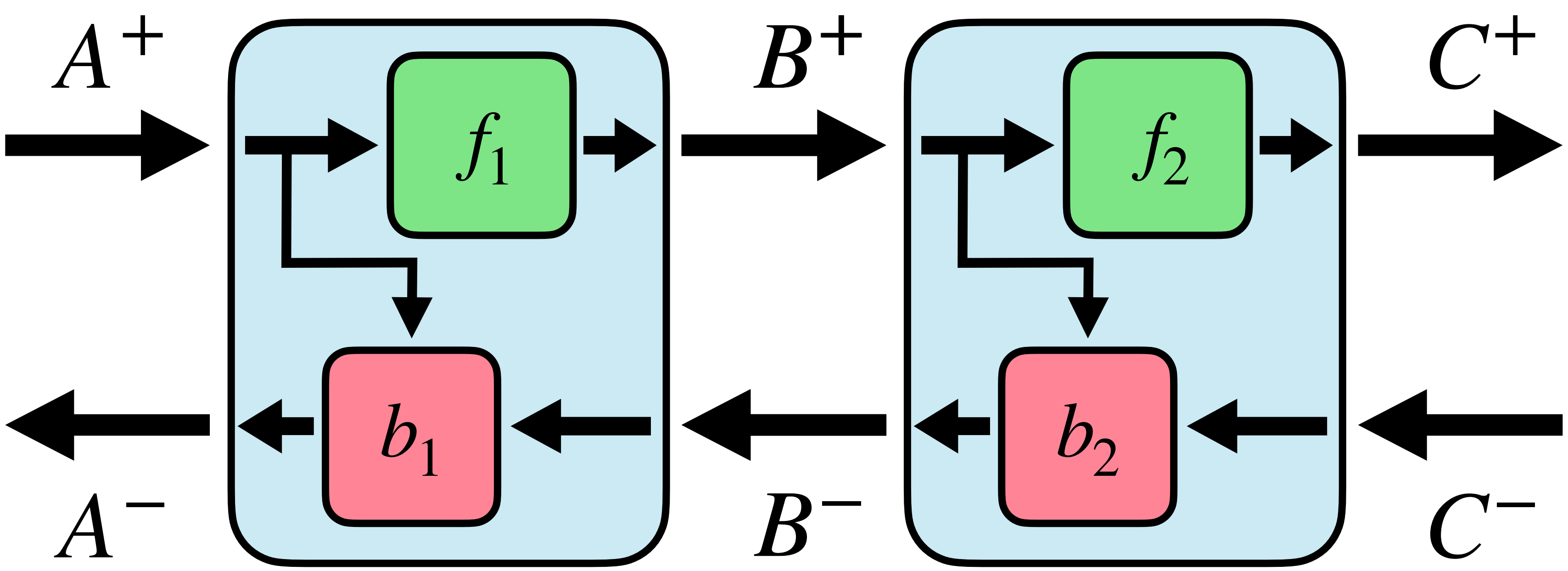


Gödel's Functional (Dialectica) Interpretation



Gödel's Functional (Dialectica) Interpretation

$$f_2 \circ f_1 : A^+ \to C^+$$



$$\lambda x^{A^+} . b_2(f_1(x)) \circ b_1(x) : A^+ \to C^- \to A^-$$

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Iterated product of selection functions.

Over some basic system, arithmetical induction

$$\mathbf{IND} \quad : \quad \forall n^{\mathbb{N}}(\forall i < n\, A(i) \to A(n)) \to \forall n^{\mathbb{N}} A(n)$$

is equivalent to finite choice

$$\mathbf{FAC} \quad : \quad \forall i < n\, \exists k\, A(i,k) \to \exists s \forall i < n\, A(i, s_i)$$

which has negative translation

$$\forall i < n \neg\neg \exists k\, A(i,k) \to \neg\neg \exists s \forall i < n\, A(i, s_i)$$

If $\exists k A(i,k)$ has interpretation $(A^+, A^-, |A|^x_y)$, then premise assumes family of (local) selection functions

$$\varepsilon_i \colon (A^+ \to A^-) \to A^+$$

while conclusion requires a (global) selection function

$$(\Pi_{i<n} A^+ \to A^-) \to \Pi_{i<n} A^+$$

This can be done by iterating product of selection fcts $\otimes \colon J_R X \times J_R Y \to J_R(X \times Y)$.

Arithmetic

M. Escardó and P. Oliva
Selection functions, bar recursion and backward induction
Mathematical Structures in Computer Science, 20(2):127-168, 2010
Connection between game theory and finite prod. of sel. fcts.

M. Escardó, P. Oliva and T. Powell
System T and the product of selection functions
Computer Science Logic, 2011
Induction, bounded collection and finite choice via prod. of sel. fcts.

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Analysis

The Principles.

$$\mathbf{LEM} : \forall n^{\mathbb{N}}(A_n \vee \neg A_n)$$

$$\mathbf{CA} : \exists f^{\mathbb{N}\to\mathbb{B}} \forall n^{\mathbb{N}}(f(n) \leftrightarrow A_n)$$

$$\mathbf{AC}_0 : \forall n^{\mathbb{N}} \exists x^{\sigma} A_n(x) \to \exists \alpha^{\sigma^{\mathbb{N}}} \forall n\, A_n(\alpha_n)$$

$$\mathbf{DNS} : \forall n^{\mathbb{N}} \neg\neg A_n \to \neg\neg \forall n^{\mathbb{N}} A_n$$

Spector'62.

- If $\mathbf{LEM} \vdash A$ then $\vdash A^N$ (negative translation)
- $\mathbf{LEM} + \mathbf{AC}_0 \vdash \mathbf{CA}$
- $\mathbf{DNS} + \neg\neg\mathbf{AC}_0 \vdash (\mathbf{AC}_0)^N$
- Bar recursion solves Dialectica interpretation of **DNS**

Interpreting AC_0 (classically)

Problem A (solves challenge).

Given $\varepsilon_n^{(\sigma\to\tau)\to\sigma}$, $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau}$ find $\alpha^{\sigma^{\mathbb{N}}}$, $n^{\mathbb{N}}$ and $p^{\sigma\to\tau}$ such that

$$A_n(\varepsilon_n(p), p(\varepsilon_n(p))) \rightarrow A_m(\alpha_m, q(\alpha))$$

where $m = \omega(\alpha)$.

global selection

Problem B (solution for B gives solution for A).

Given $\varepsilon_n^{(\sigma\to\tau)\to\sigma}$, $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau}$ find sequences $\alpha^{\sigma^{\mathbb{N}}}$ and $p_n^{\sigma\to\tau}$ such that

local selections

$$\alpha_n = \varepsilon_n(p_n)$$
$$p_n(\alpha_n) = q(\alpha)$$

for $n \leq \omega(\alpha)$.

Interpreting $\mathbf{AC}_0$ (classically)

Problem B (solution for B gives solution for A).

Given $\varepsilon_n^{(\sigma\to\tau)\to\sigma}$, $\omega^{\sigma^{\mathbb{N}}\to\mathbb{N}}$ and $q^{\sigma^{\mathbb{N}}\to\tau}$ find sequences $\alpha^{\sigma^{\mathbb{N}}}$ and $p_n^{\sigma\to\tau}$ such that

$$\alpha_n = \varepsilon_n(p_n)$$
$$p_n(\alpha_n) = q(\alpha)$$

for $n \leq \omega(\alpha)$.

Solution for B (bar recursion).

Define $B(s)\colon\ \sigma^{\mathbb{N}}$ as follows $(s\colon\ \sigma^*$ and $\hat{s} = s * 0\ldots)$

$$B(s) = \begin{cases} \hat{s} & \text{if } \omega(\hat{s}) < |s| \\ B(s * c) & \text{if } \omega(\hat{s}) \geq |s| \end{cases}$$

where $c =_\sigma \varepsilon_{|s|}(\kappa_s)$ and $\kappa_s(x) =_\tau q(B(s * x))$.

Take $\alpha = B(\langle\ \rangle)$ and $p_n = \kappa_{\overline{\alpha}(n)}$, where $\overline{\alpha}(n) = \langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle$.

iterated product of selection function

Bar recursion

P. Oliva
Understanding and using Spector's bar recursive interpretation of classical analysis
CiE'2006, LNCS 3988:423-434, Springer, 2006
Some simple applications of Spector's bar recursion

M. Escardó and P. Oliva
Bar recursion and products of selection functions
The Journal of Symbolic Logic, 80(1):1-28, 2015
Equivalence between bar recursion and countable prod. of sel. fcts.

M. Escardó and P. Oliva
The Herbrand functional interpretation of the double negation shift
The Journal of Symbolic Logic, 82(2):590-607, 2017
The selection monad transformer...