A Modular Unifying View of Proof Translations (Part II)

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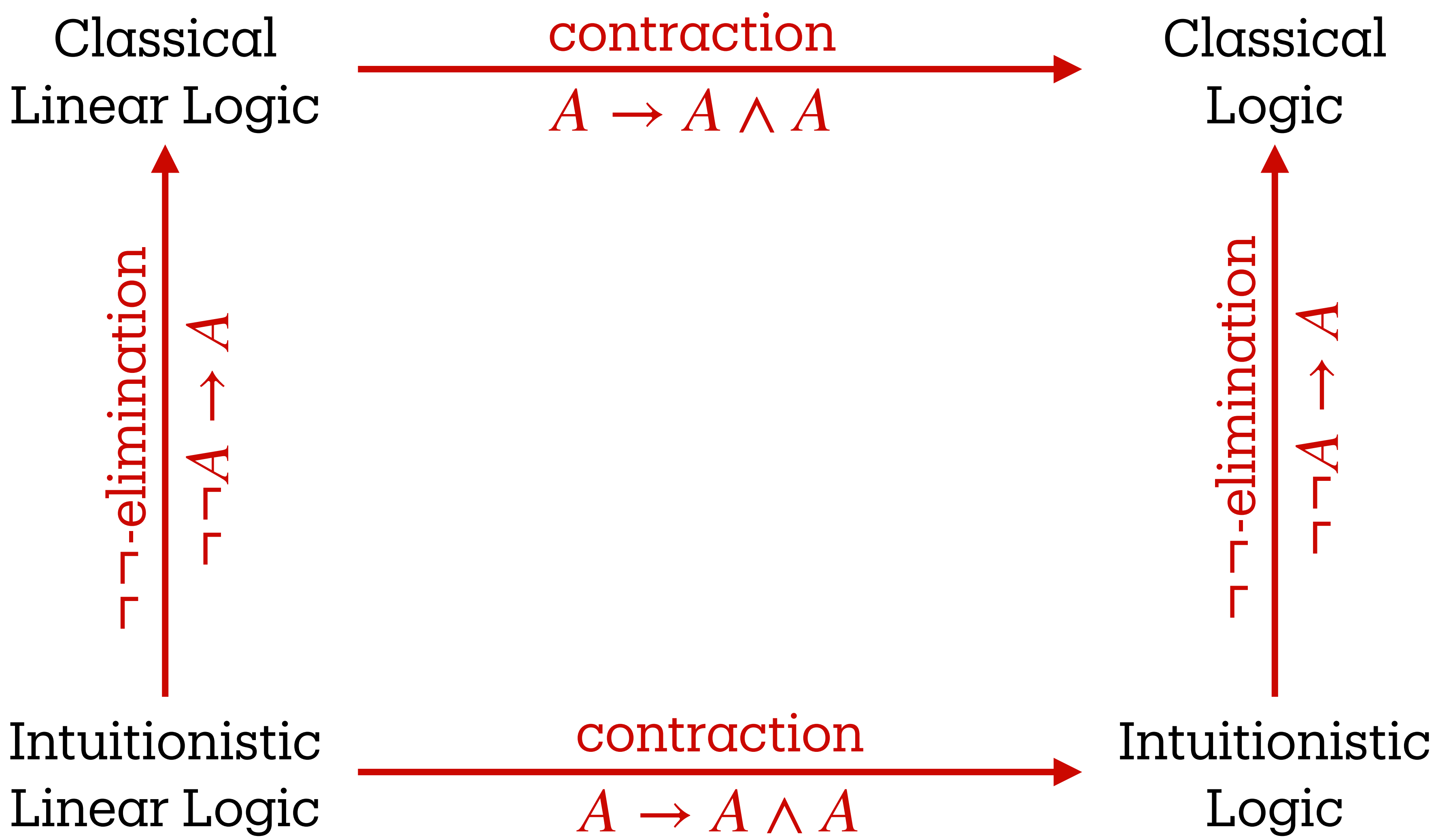
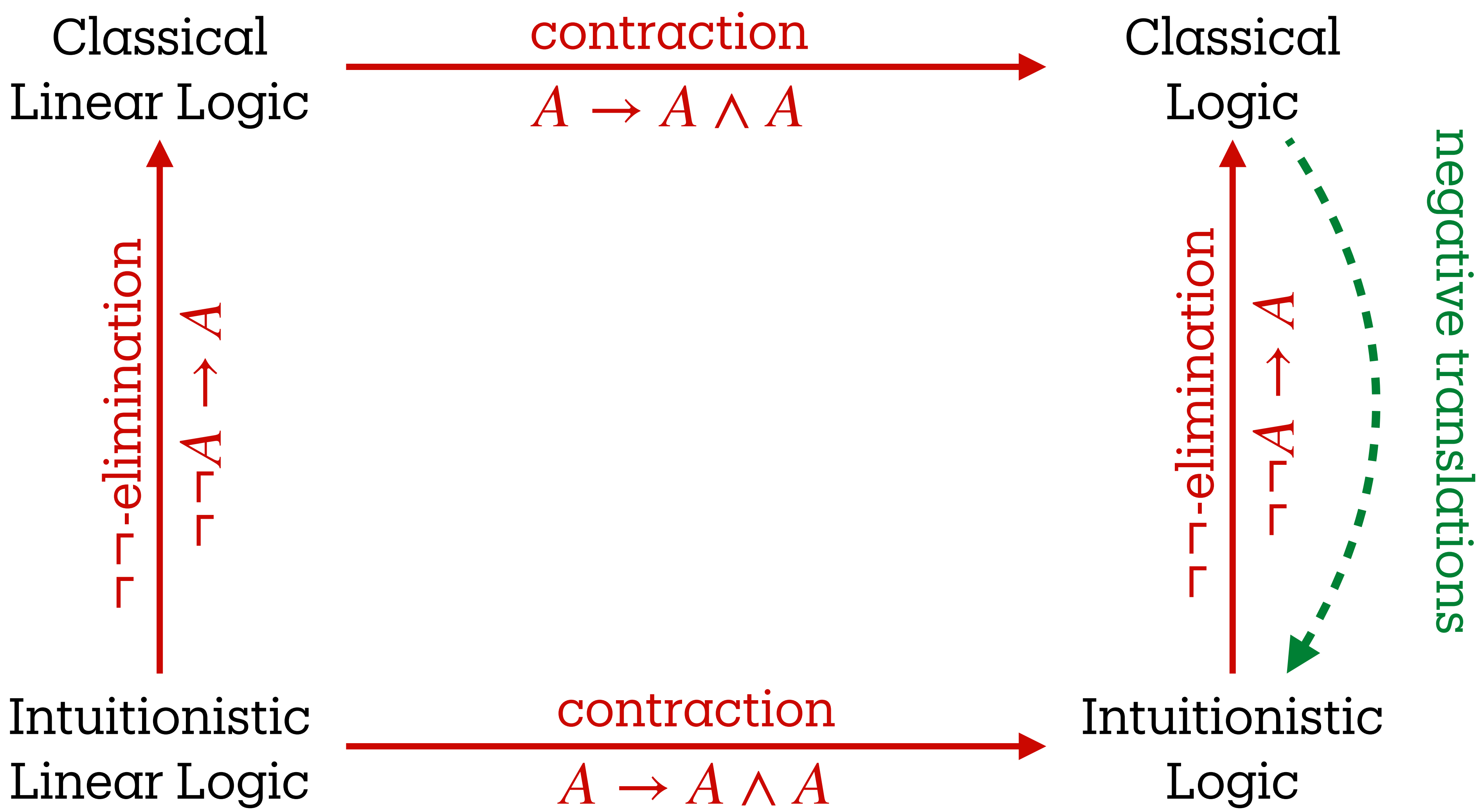
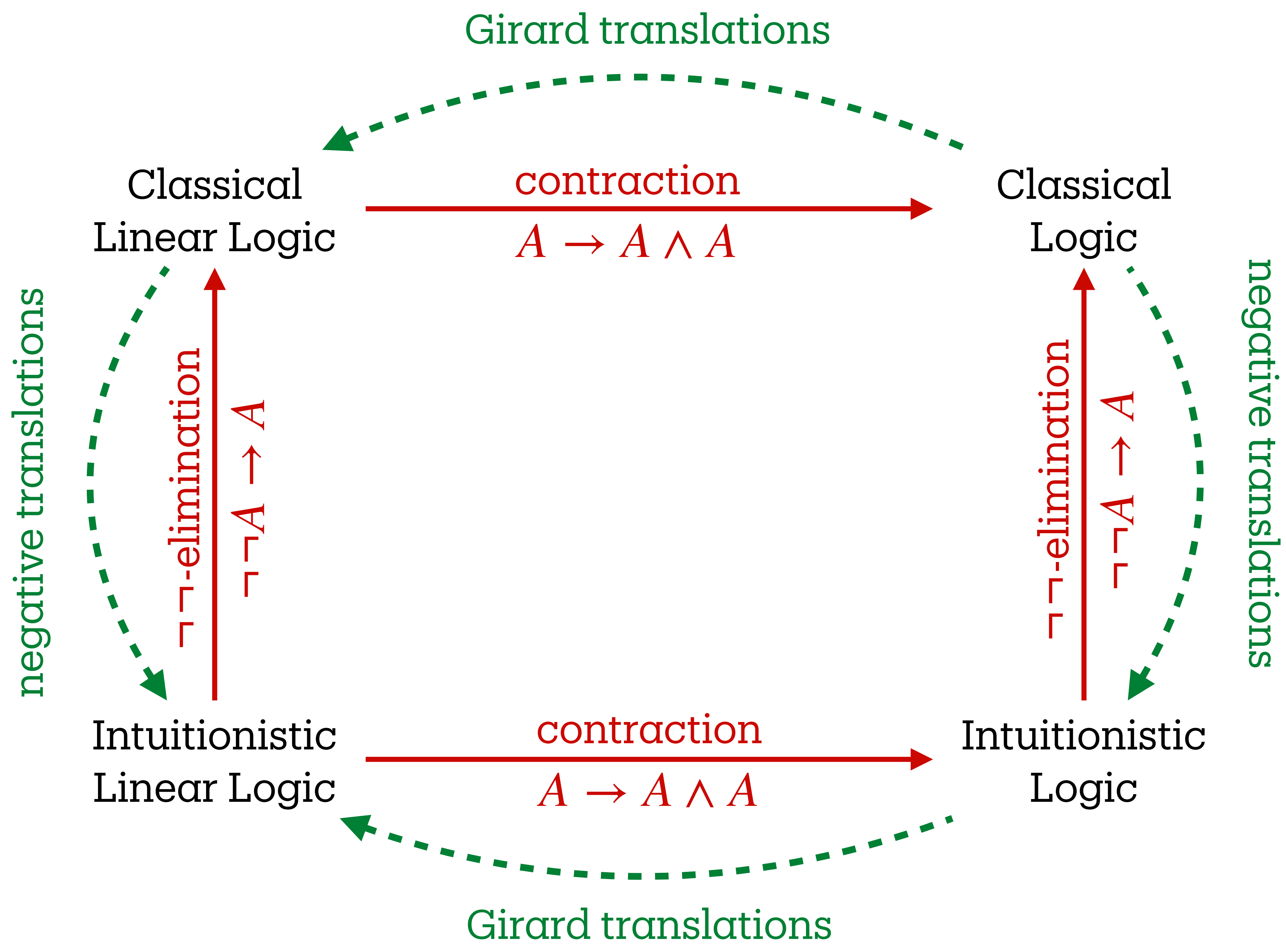

Classical
Linear Logic
contraction
A → A ∧ A
Classical
Logic
¬¬-elimination
¬¬A → A
¬¬-elimination
¬¬A → A
Intuitionistic
Linear Logic
contraction
A → A ∧ A
Intuitionistic
Logic



Classical
Linear Logic
contraction
$A \to A \wedge A$
Classical
Logic
negative translations
¬¬-elimination
$\neg\neg A \to A$
¬¬-elimination
$\neg\neg A \to A$
Intuitionistic
Linear Logic
contraction
$A \to A \wedge A$
Intuitionistic
Logic


Girard translations
Classical
Linear Logic
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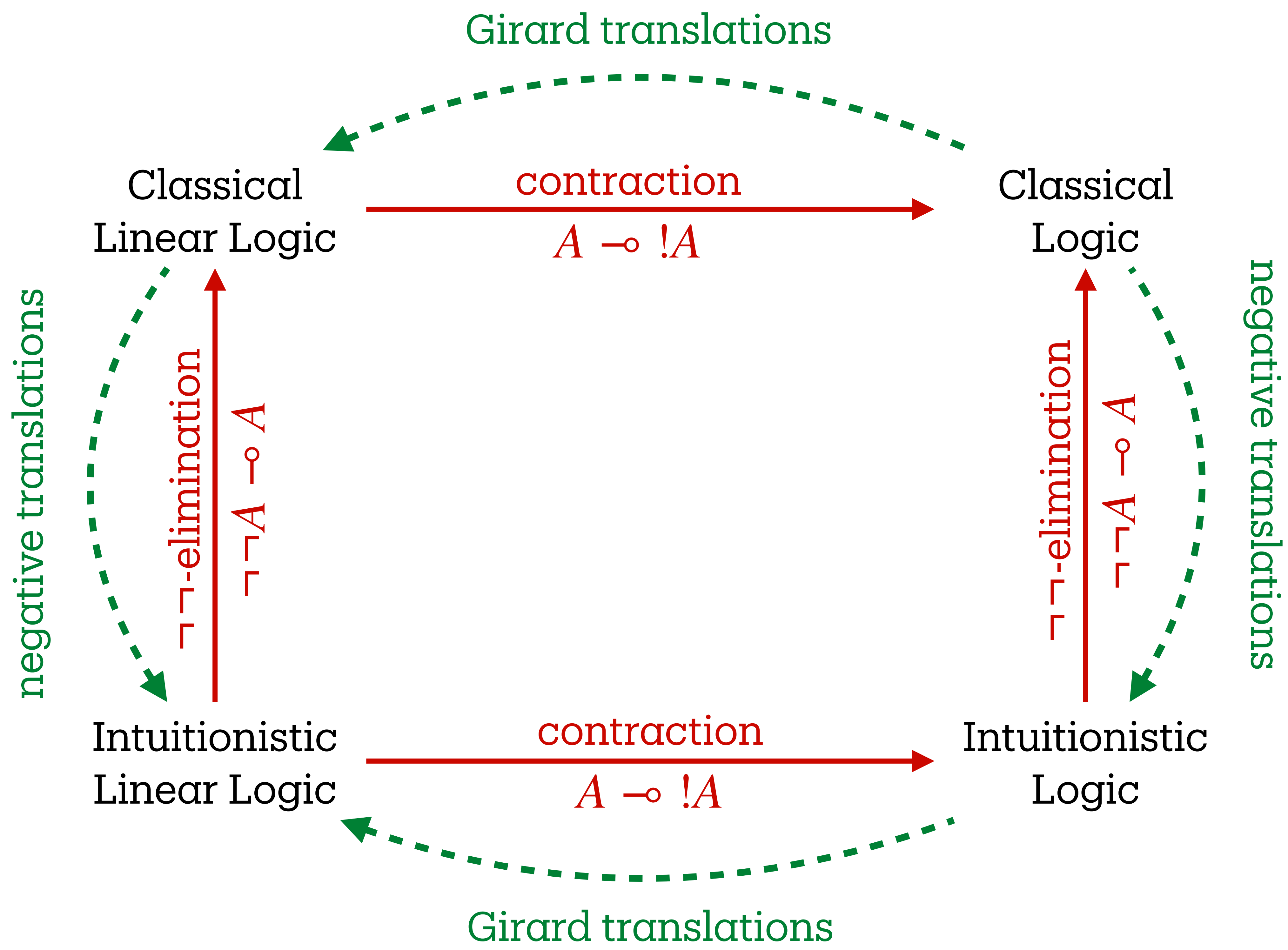
Girard translations
Classical Linear Logic
contraction
A ⊸ !A
Classical Logic
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contraction
A ⊸ !A
Intuitionistic Logic
Girard translations

Intuitionistic Linear Logic (as basis system)

Intuitionistic Linear Logic

$$\frac{}{A \vdash A}\,(\mathbf{id}) \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\,(\mathbf{cut})$$

$$\frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}\,(1\mathbf{L}) \qquad \frac{}{\vdash 1}\,(1\mathbf{R}) \qquad \frac{}{\Gamma \vdash \top}\,(\top\mathbf{R}) \qquad \frac{}{\Gamma, 0 \vdash C}\,(0\mathbf{L})$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes\mathbf{R}) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,(\otimes\mathbf{L})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\multimap\mathbf{R}) \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,(\multimap\mathbf{L})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,(\&\mathbf{R}) \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \,\&\, B \vdash C}\,(\&\mathbf{L}) \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C}\,(\&\mathbf{L})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\,(\oplus\mathbf{R}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\,(\oplus\mathbf{R}) \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,(\oplus\mathbf{L})$$

Intuitionistic Linear Logic

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\ (\forall R) \qquad \frac{\Gamma, A[t/x] \vdash C}{\Gamma, \forall x A \vdash C}\ (\forall L)$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ (\exists R) \qquad \frac{\Gamma, A \vdash C}{\Gamma, \exists x A \vdash C}\ (\exists L)$$

$$\frac{\Gamma, !A, !A \vdash C}{\Gamma, !A \vdash C}\ (\text{con}) \qquad \frac{\Gamma \vdash C}{\Gamma, !A \vdash C}\ (\text{wkn}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!R) \qquad \frac{\Gamma, A \vdash C}{\Gamma, !A \vdash C}\ (!L)$$

To ILL we will consider adding the following axiom schema:

(PRO) $A \vdash !A$

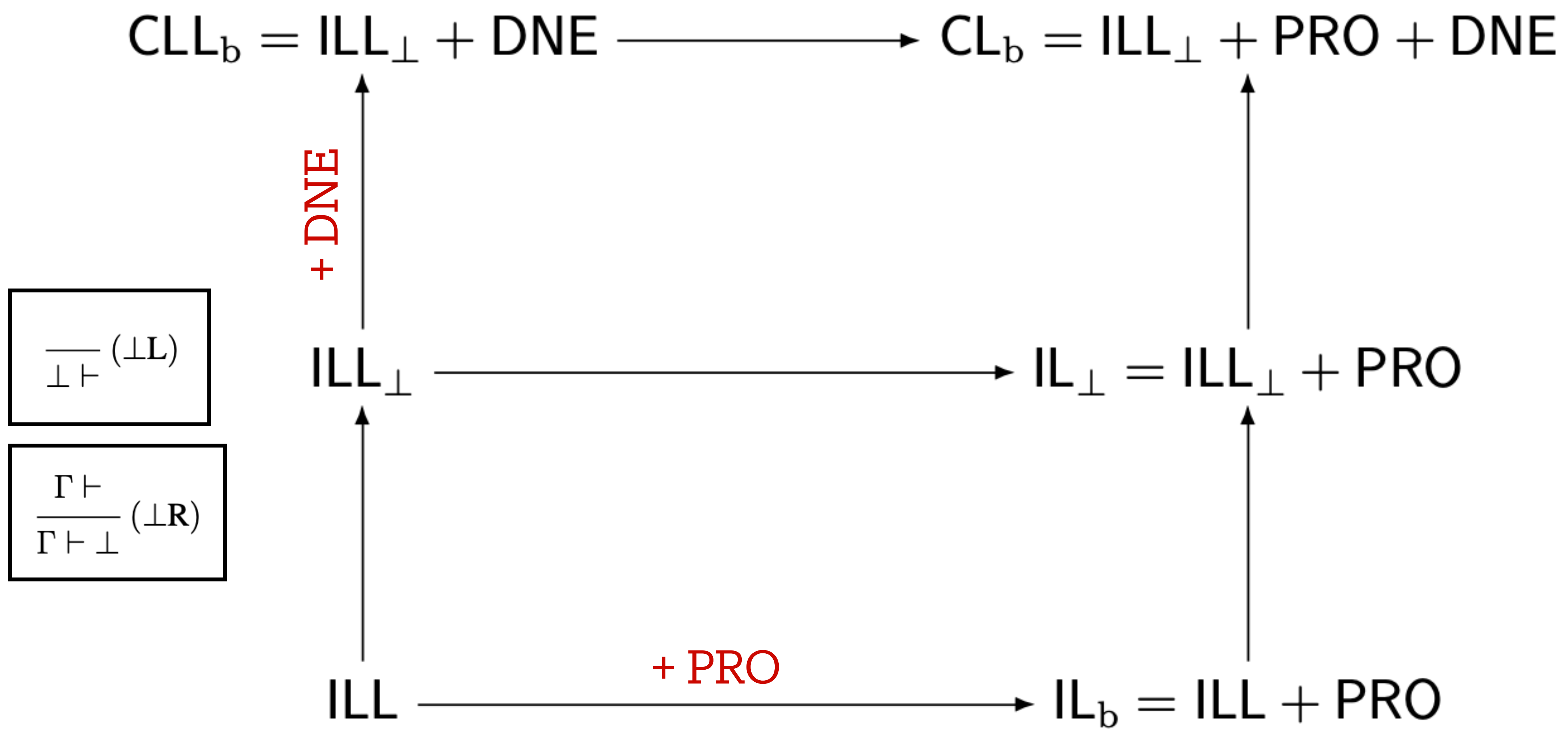
(DNE) $\neg\neg A \vdash A$.

Definition 2 (Translating $\mathcal{L}(\mathsf{CLL})$ into $\mathcal{L}(\mathsf{ILL}_\bot)$). *The translation $A \mapsto A^\ddagger$ is defined inductively as follows:*

$$
\begin{array}{rcl@{\qquad}rcl}
(A \otimes B)^\ddagger & :\equiv & A^\ddagger \otimes B^\ddagger & P^\ddagger & :\equiv & P \quad (P \text{ atomic}) \\
(A \,\&\, B)^\ddagger & :\equiv & A^\ddagger \,\&\, B^\ddagger & (\forall x A)^\ddagger & :\equiv & \forall x A^\ddagger \\
(A \oplus B)^\ddagger & :\equiv & A^\ddagger \oplus B^\ddagger & (\exists x A)^\ddagger & :\equiv & \exists x A^\ddagger \\
(A \multimap B)^\ddagger & :\equiv & A^\ddagger \multimap B^\ddagger & (!A)^\ddagger & :\equiv & !A^\ddagger \\
(A \parr B)^\ddagger & :\equiv & \neg(\neg A^\ddagger \otimes \neg B^\ddagger) & (?A)^\ddagger & :\equiv & \neg!\neg A^\ddagger
\end{array}
$$

Proposition 2 (CLL as an extension of $\mathsf{ILL}_\bot$). *Let* $\mathsf{CLL}_\mathrm{b} := \mathsf{ILL}_\bot + \mathsf{DNE}$. *Then* $\vdash_{\mathsf{CLL}} A$ *iff* $\vdash_{\mathsf{CLL}_\mathrm{b}} A^\ddagger$.

ILL as basis system



Some Theorems of ILL

Proposition 4. *The following* $\mathsf{CLL_b}$ *(and hence also* $\mathsf{CL_b}$*) equivalences hold / fail in* $\mathsf{IL}_\bot$ *and* $\mathsf{ILL}_\bot$*:*

				$\mathsf{IL}_\bot$	$\mathsf{ILL}_\bot$
(i)	$\neg\neg(\neg\neg A \otimes \neg\neg B)$	$\multimap\!\circ$	$\neg\neg(A \otimes B)$	✓	✓
(ii)	$\neg\neg(\neg\neg A \,\&\, \neg\neg B)$	$\multimap\!\circ$	$\neg\neg(A \,\&\, B)$	✓	×
(iii)	$\neg\neg(\neg\neg A \oplus \neg\neg B)$	$\multimap\!\circ$	$\neg\neg(A \oplus B)$	✓	✓
(iv)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimap\!\circ$	$\neg\neg(A \multimap B)$	✓	×
(v)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimap\!\circ$	$\neg\neg(A \multimap \neg\neg B)$	✓	✓
(vi)	$\neg\neg\forall x \neg\neg A$	$\multimap\!\circ$	$\neg\neg\forall x A$	×	×
(vii)	$\neg\neg\exists x \neg\neg A$	$\multimap\!\circ$	$\neg\neg\exists x A$	✓	✓
$(viii)$	$\neg\neg!\neg\neg A$	$\multimap\!\circ$	$\neg\neg!A$	✓	×
(ix)	$\neg\neg(\neg\neg A \otimes \neg\neg B)$	$\multimap\!\circ$	$\neg\neg A \otimes \neg\neg B$	✓	×
(x)	$\neg\neg(\neg\neg A \,\&\, \neg\neg B)$	$\multimap\!\circ$	$\neg\neg A \,\&\, \neg\neg B$	✓	✓
(xi)	$\neg\neg(\neg\neg A \oplus \neg\neg B)$	$\multimap\!\circ$	$\neg\neg A \oplus \neg\neg B$	×	×
(xii)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimap\!\circ$	$\neg\neg A \multimap \neg\neg B$	✓	✓
$(xiii)$	$\neg\neg\forall x \neg\neg A$	$\multimap\!\circ$	$\forall x \neg\neg A$	✓	✓
(xiv)	$\neg\neg\exists x \neg\neg A$	$\multimap\!\circ$	$\exists x \neg\neg A$	×	×
(xv)	$\neg\neg!\neg\neg A$	$\multimap\!\circ$	$!\neg\neg A$	✓	×

simplify 'from inside'

Proposition 4. *The following* $\mathsf{CLL_b}$ *(and hence also* $\mathsf{CL_b}$*) equivalences hold / fail in* $\mathsf{IL}_\perp$ *and* $\mathsf{ILL}_\perp$*:*

				$\mathsf{IL}_\perp$	$\mathsf{ILL}_\perp$
(i)	$\neg\neg(\neg\neg A \otimes \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg(A \otimes B)$	✓	✓
(ii)	$\neg\neg(\neg\neg A \,\&\, \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg(A \,\&\, B)$	✓	×
(iii)	$\neg\neg(\neg\neg A \oplus \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg(A \oplus B)$	✓	✓
(iv)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg(A \multimap B)$	✓	×
(v)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg(A \multimap \neg\neg B)$	✓	✓
(vi)	$\neg\neg\forall x \neg\neg A$	$\multimap\!\!\circ$	$\neg\neg\forall x A$	×	×
(vii)	$\neg\neg\exists x \neg\neg A$	$\multimap\!\!\circ$	$\neg\neg\exists x A$	✓	✓
(viii)	$\neg\neg!\neg\neg A$	$\multimap\!\!\circ$	$\neg\neg!A$	✓	×
(ix)	$\neg\neg(\neg\neg A \otimes \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg A \otimes \neg\neg B$	✓	×
(x)	$\neg\neg(\neg\neg A \,\&\, \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg A \,\&\, \neg\neg B$	✓	✓
(xi)	$\neg\neg(\neg\neg A \oplus \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg A \oplus \neg\neg B$	×	×
(xii)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimap\!\!\circ$	$\neg\neg A \multimap \neg\neg B$	✓	✓
(xiii)	$\neg\neg\forall x \neg\neg A$	$\multimap\!\!\circ$	$\forall x \neg\neg A$	✓	✓
(xiv)	$\neg\neg\exists x \neg\neg A$	$\multimap\!\!\circ$	$\exists x \neg\neg A$	×	×
(xv)	$\neg\neg!\neg\neg A$	$\multimap\!\!\circ$	$!\neg\neg A$	✓	×

Proposition 4. *The following* $\mathsf{CLL_b}$ *(and hence also* $\mathsf{CL_b}$*) equivalences hold / fail in* $\mathsf{IL_\bot}$ *and* $\mathsf{ILL_\bot}$*:*

					$\mathsf{IL_\bot}$	$\mathsf{ILL_\bot}$
simplify 'from inside'	*(i)*	$\neg\neg(\neg\neg A \otimes \neg\neg B)$	$\multimapboth$	$\neg\neg(A \otimes B)$	✓	✓
	(ii)	$\neg\neg(\neg\neg A \,\&\, \neg\neg B)$	$\multimapboth$	$\neg\neg(A \,\&\, B)$	✓	×
	(iii)	$\neg\neg(\neg\neg A \oplus \neg\neg B)$	$\multimapboth$	$\neg\neg(A \oplus B)$	✓	✓
	(iv)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimapboth$	$\neg\neg(A \multimap B)$	✓	×
	(v)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimapboth$	$\neg\neg(A \multimap \neg\neg B)$	✓	✓
	(vi)	$\neg\neg\forall x \neg\neg A$	$\multimapboth$	$\neg\neg\forall x A$	×	×
	(vii)	$\neg\neg\exists x \neg\neg A$	$\multimapboth$	$\neg\neg\exists x A$	✓	✓
	(viii)	$\neg\neg!\neg\neg A$	$\multimapboth$	$\neg\neg!A$	✓	×
simplify 'from outside'	*(ix)*	$\neg\neg(\neg\neg A \otimes \neg\neg B)$	$\multimapboth$	$\neg\neg A \otimes \neg\neg B$	✓	×
	(x)	$\neg\neg(\neg\neg A \,\&\, \neg\neg B)$	$\multimapboth$	$\neg\neg A \,\&\, \neg\neg B$	✓	✓
	(xi)	$\neg\neg(\neg\neg A \oplus \neg\neg B)$	$\multimapboth$	$\neg\neg A \oplus \neg\neg B$	×	×
	(xii)	$\neg\neg(\neg\neg A \multimap \neg\neg B)$	$\multimapboth$	$\neg\neg A \multimap \neg\neg B$	✓	✓
	(xiii)	$\neg\neg\forall x \neg\neg A$	$\multimapboth$	$\forall x \neg\neg A$	✓	✓
	(xiv)	$\neg\neg\exists x \neg\neg A$	$\multimapboth$	$\exists x \neg\neg A$	×	×
	(xv)	$\neg\neg!\neg\neg A$	$\multimapboth$	$!\neg\neg A$	✓	×

Proposition 5. *The following* $\mathsf{IL_b} = \mathsf{ILL} + \mathsf{PRO}$ *equivalences hold / fail in* ILL^4*:*

					ILL
simplify 'from outside'	(i)	$!(!A \otimes !B)$	$\multimapboth$	$!A \otimes !B$	✓
	(ii)	$!(!A \,\&\, !B)$	$\multimapboth$	$!A \,\&\, !B$	×
	(iii)	$!(!A \oplus !B)$	$\multimapboth$	$!A \oplus !B$	✓
	(iv)	$!(!A \multimap !B)$	$\multimapboth$	$!A \multimap !B$	×
	(v)	$!\forall x !A$	$\multimapboth$	$\forall x !A$	×
	(vi)	$!\exists x !A$	$\multimapboth$	$\exists x !A$	✓
	(vii)	$!!!A$	$\multimapboth$	$!!A$	✓
simplify 'from inside'	$(viii)$	$!(!A \otimes !B)$	$\multimapboth$	$!(A \otimes B)$	×
	(ix)	$!(!A \,\&\, !B)$	$\multimapboth$	$!(A \,\&\, B)$	✓
	(x)	$!(!A \multimap !B)$	$\multimapboth$	$!(A \multimap B)$	×
	(xi)	$!(!A \multimap !B)$	$\multimapboth$	$!(!A \multimap B)$	✓
	(xii)	$!\forall x !A$	$\multimapboth$	$!\forall x A$	✓
	$(xiii)$	$!\exists x !A$	$\multimapboth$	$!\exists x A$	×

G. Ferreira, P. Oliva and C. L. Protin
On the various translations between classical, intuitionistic and linear logic
Preprint — https://arxiv.org/abs/2409.02249

Simplifying Proof Translations

Translations of CL into IL (negative translations)

Definition (Kolmogorov, 1925).

The Kolmogorov translation (*outer presentation*) is inductively defined as:

$$(A \otimes B)^{K_o} :\equiv \neg\neg(A^{K_o} \otimes B^{K_o}) \qquad (P)^{K_o} :\equiv \neg\neg P$$

$$(A \,\&\, B)^{K_o} :\equiv \neg\neg(A^{K_o} \,\&\, B^{K_o}) \qquad (\forall x A)^{K_o} :\equiv \neg\neg\forall x A^{K_o}$$

$$(A \oplus B)^{K_o} :\equiv \neg\neg(A^{K_o} \oplus B^{K_o}) \qquad (\exists x A)^{K_o} :\equiv \neg\neg\exists x A^{K_o}$$

$$(A \multimap B)^{K_o} :\equiv \neg\neg(A^{K_o} \multimap B^{K_o}) \qquad (!A)^{K_o} :\equiv \neg\neg !A^{K_o}$$

Theorem (Kolmogorov, 1925).

If A is provable in CL then A^{K_o} is provable in IL

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$$(A \,\&\, B)^{K_o} :\equiv \neg\neg(A^{K_o} \,\&\, B^{K_o}) \qquad (\forall x A)^{K_o} :\equiv \neg\neg \forall x A^{K_o}$$
$$(A \oplus B)^{K_o} :\equiv \neg\neg(A^{K_o} \oplus B^{K_o}) \qquad (\exists x A)^{K_o} :\equiv \neg\neg \exists x A^{K_o}$$
$$(A \multimap B)^{K_o} :\equiv \neg\neg(A^{K_o} \multimap B^{K_o}) \qquad (!A)^{K_o} :\equiv \neg\neg !A^{K_o}$$

Theorem (Kolmogorov, 1925).

If A is provable in CL then A^{K_o} is provable in IL

Examples.

- $(P \otimes Q \multimap !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg !\neg\neg R)$
- $(\forall x P \multimap \exists x Q)^{K_o} \equiv \neg\neg(\neg\neg \forall x \neg\neg P \multimap \neg\neg \exists x \neg\neg Q)$

Simplifying the Kolmogorov translation

$$(P \otimes Q \multimap !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg!\neg\neg R)$$

Simplifying the Kolmogorov translation

$$(P \otimes Q \multimap\, !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg!\neg\neg R)$$

$$\Updownarrow \quad \text{simplifying ‘from inside’}$$

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!\neg\neg R)$$

Simplifying the Kolmogorov translation

$$(P \otimes Q \multimap\, !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

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Simplifying the Kolmogorov translation

$$(P \otimes Q \multimap\ !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg!\neg\neg R)$$

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$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(P \otimes Q \multimap\ !R) \quad \equiv \quad (P \otimes Q \multimap\ !R)^{Ku}$$

Simplifying the Kolmogorov translation

$$(P \otimes Q \multimap\ !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(P \otimes Q \multimap\ !R) \quad \equiv \quad (P \otimes Q \multimap\ !R)^{Ku}$$

Kuroda translation

Simplifying the Kolmogorov translation

$$(P \otimes Q \multimap\ !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!R)$$

valid in IL but not in ILL →

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(P \otimes Q \multimap\ !R) \quad \equiv \quad (P \otimes Q \multimap\ !R)^{Ku}$$

Kuroda translation

Translations of IL into ILL (Girard translations)

Definition (Girard, 1987).

The Girard ‘full’ translation (*outer presentation*) is defined as:

$$
\begin{aligned}
(A \otimes B)^{Gf} &:\equiv\ !(A^{Gf} \otimes B^{Gf}) & \qquad (P)^{Gf} &:\equiv\ !P \\
(A \mathbin{\&} B)^{Gf} &:\equiv\ !(A^{Gf} \mathbin{\&} B^{Gf}) & (\forall x A)^{Gf} &:\equiv\ !\forall x A^{Gf} \\
(A \oplus B)^{Gf} &:\equiv\ !(A^{Gf} \oplus B^{Gf}) & (\exists x A)^{Gf} &:\equiv\ !\exists x A^{Gf} \\
(A \multimap B)^{Gf} &:\equiv\ !(A^{Gf} \multimap B^{Gf}) & (!A)^{Gf} &:\equiv\ !!A^{Gf}
\end{aligned}
$$

Theorem (Girard, 1987).

If A is provable in IL then A^{Gf} is provable in ILL

Translations of IL into ILL (Girard translations)

Definition (Girard, 1987).

The Girard ‘full’ translation (*outer presentation*) is defined as:

$$(A \otimes B)^{Gf} :\equiv\ !(A^{Gf} \otimes B^{Gf}) \qquad (P)^{Gf} :\equiv\ !P$$
$$(A \mathbin{\&} B)^{Gf} :\equiv\ !(A^{Gf} \mathbin{\&} B^{Gf}) \qquad (\forall x A)^{Gf} :\equiv\ !\forall x A^{Gf}$$
$$(A \oplus B)^{Gf} :\equiv\ !(A^{Gf} \oplus B^{Gf}) \qquad (\exists x A)^{Gf} :\equiv\ !\exists x A^{Gf}$$
$$(A \multimap B)^{Gf} :\equiv\ !(A^{Gf} \multimap B^{Gf}) \qquad (!A)^{Gf} :\equiv\ !!A^{Gf}$$

Theorem (Girard, 1987).

If A is provable in IL then A^{Gf} is provable in ILL

Examples.

- $(P \mathbin{\&} Q \multimap !R)^{Gf} \equiv\ !(!(!P \mathbin{\&} !Q) \multimap !!!R)$
- $(\forall x P \multimap \exists x Q)^{Gf} \equiv\ !(!\forall x !P \multimap !\exists x !Q)$

Simplifying Girard's 'full' translation

$$(P \,\&\, Q \multimap\, !R)^{Gf} \equiv\, !(\,!(\,!P \& \,!Q) \multimap\, !\,!\,!R)$$

Simplifying Girard's 'full' translation

$$(P \mathbin{\&} Q \multimap\, !R)^{Gf} \equiv\, !(\,!(\,!P \mathbin{\&}\, !Q) \multimap\, !\,!\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \mathbin{\&} Q) \multimap\, !\,!\,!R)$$

Simplifying Girard's 'full' translation

$$(P \,\&\, Q \multimap\, !R)^{Gf} \equiv\, !(\,!(\,!P \& \,!Q) \multimap\, !\,!\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \,\&\, Q) \multimap\, !\,!\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \,\&\, Q) \multimap\, !\,!R)$$

Simplifying Girard's 'full' translation

$$(P \mathbin{\&} Q \multimap\ !R)^{Gf} \equiv\ !(\,!(\,!P \mathbin{\&}\, !Q) \multimap\ !\,!\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \mathbin{\&} Q) \multimap\ !\,!\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \mathbin{\&} Q) \multimap\ !\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \mathbin{\&} Q) \multimap\ !R) \quad \equiv \quad (P \mathbin{\&} Q \multimap\ !R)^{\circ}$$

Simplifying Girard's 'full' translation

$$(P \mathbin{\&} Q \multimap\, !R)^{Gf} \equiv\, !(\,!(\,!P \mathbin{\&}\, !Q) \multimap\, !\,!\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \mathbin{\&} Q) \multimap\, !\,!\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \mathbin{\&} Q) \multimap\, !\,!R)$$

$\Updownarrow$ simplifying 'from inside'

$$!(\,!(P \mathbin{\&} Q) \multimap\, !R) \quad \equiv \quad (P \mathbin{\&} Q \multimap\, !R)^{\circ}$$

call-by-name translation

Kolmogorov translation (linear)

$$(P \otimes Q \multimap\, !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying ‘from inside’

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying ‘from inside’

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!R)$$

$\Updownarrow$ simplifying ‘from inside’

$$\neg\neg(P \otimes Q \multimap \neg\neg!R) \quad \equiv \quad (P \otimes Q \multimap\, !R)^{Ku}$$

Kolmogorov translation (linear)

$$(P \otimes Q \multimap\, !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg !\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg !\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg !R)$$

need to 'leave' double negation at conclusion → $\Updownarrow$ simplifying 'from inside'

$$\neg\neg(P \otimes Q \multimap \neg\neg !R) \quad \equiv \quad (P \otimes Q \multimap\, !R)^{Ku}$$

Kolmogorov translation (linear)

$$(P \otimes Q \multimap\, !R)^{K_o} \equiv \neg\neg(\neg\neg(\neg\neg P \otimes \neg\neg Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!\neg\neg R)$$

$\Updownarrow$ simplifying 'from inside'

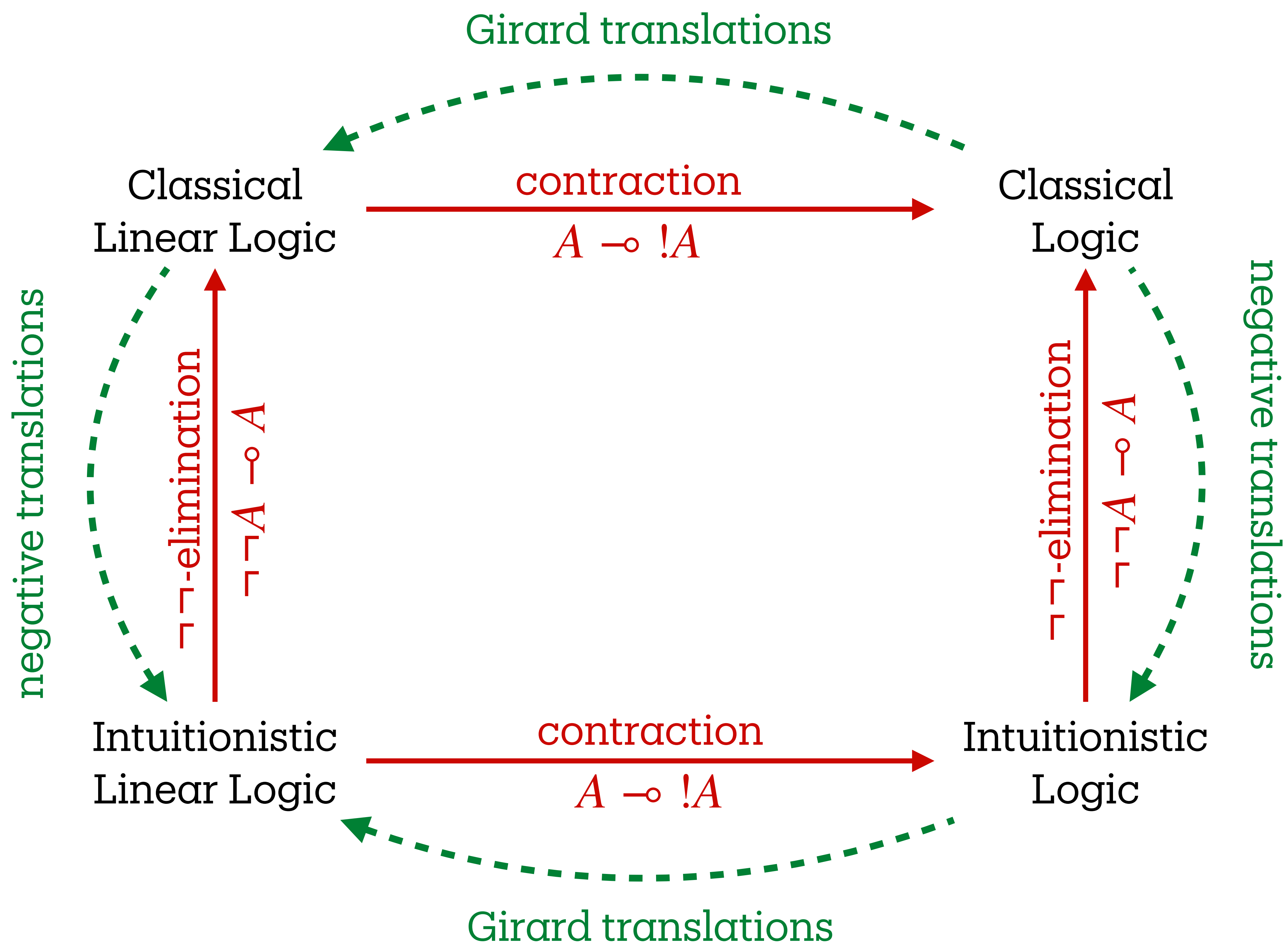
$$\neg\neg(\neg\neg(P \otimes Q) \multimap \neg\neg!R)$$

need to 'leave' double negation at conclusion

$\Updownarrow$ simplifying 'from inside'

$$\neg\neg(P \otimes Q \multimap \neg\neg!R) \quad \equiv \quad (P \otimes Q \multimap\, !R)^{Ku}$$

Kuroda translation (linear)


Girard translations
Classical Linear Logic
contraction
$A \multimap !A$
Classical Logic
negative translations
¬¬-elimination
$\neg\neg A \multimap A$
¬¬-elimination
$\neg\neg A \multimap A$
negative translations
Intuitionistic Linear Logic
contraction
$A \multimap !A$
Intuitionistic Logic
Girard translations

Remarks on Algebraic Semantics

- pocrims (partially ordered, commutative, integral, residuated monoids) are algebraic models of ILL
- Elements A such that $\neg\neg A = A$ are called regular
- In involutive pocrims all elements are regular (classical)
- Given any pocrim, we can define two involutive pocrims:
  - Involutive core: sub-algebra of regular elements

    Gödel-Gentzen translation
  - Involutive replica: quotient algebra of regular elements

    Kuroda translation

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Definition 2.9. *Given a bounded pocrim* $\mathbf{P}$ *we define the following structures over the signature of a bounded pocrim:*

- $\mathbf{P}^C$, *the* involutive core of $\mathbf{P}$, *is* $(P^C, \top, \bot, \hat{\cdot}, \hat{\to})$ *where* $P^C = \Im(\delta) \subseteq P$, *where* $\top$ *and* $\bot$ *are as in* P *and where* $\hat{\cdot}$ *and* $\hat{\to}$ *are defined as follows:*

$$x \mathbin{\hat{\cdot}} y \;:=\; \delta(x \cdot y)$$
$$x \mathbin{\hat{\to}} y \;:=\; x \to y$$

  We write $\iota \colon P^C \to P$ *for the inclusion.*

- $\mathbf{P}^R$, *the* involutive replica of $\mathbf{P}$ *is* $(P^R, [\top], [\bot], \check{\cdot}, \check{\to})$ *where* P^R *is the quotient* P/θ *of* P *by the equivalence relation defined by* $x \; \theta \; y$ *iff* $\delta(x) = \delta(y)$ *and where, writing* $[x]$ *for the equivalence class in* P^R *of* $x \in P$, *we define* $\check{\cdot}$ *and* $\check{\to}$ *as follows:*

$$[x] \mathbin{\check{\cdot}} [y] \;:=\; [x \cdot y]$$
$$[x] \mathbin{\check{\to}} [y] \;:=\; [x \to \delta(y)]$$

  We write $\pi \colon P \to P^R$ *for the projection.*

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Summary

- Four logics (CL, IL, ILL and CLL) as extensions of ILL
- Add DNE ($\neg\neg A \multimap A$) and/or PRO ($A \multimap\, !A$)
- Full translations add $\neg\neg$ or $!$ ‘everywhere’
- Logical equivalences allow us to simplify systematically ‘from inside’ or ‘from outside’ (*maintain modularity*)
- Kuroda is a simplification of Kolmogorov ‘from inside’ whereas Gödel-Gentzen is a simplification ‘from outside’
- Girard ‘call-by-name’ translation is a simplification from inside of full translation, while ‘call-by-value’ translation is a simplification ‘from outside’

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