Uniform Realizability Interpretations

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ON THE COMPUTATIONAL CONTENT OF THE AXIOM OF CHOICE

STEFANO BERARDI, MARC BEZEM, AND THIERRY COQUAND

Abstract. We present a possible computational content of the negative translation of classical analysis with the Axiom of (countable) Choice. Interestingly, this interpretation uses a refinement of the realizability semantics of the absurdity proposition, which is *not* interpreted as the empty type here. We also show how to compute witnesses from proofs in classical analysis of $\exists$-statements and how to extract algorithms from proofs of $\forall\exists$-statements. Our interpretation seems computationally more direct than the one based on Gödel's Dialectica interpretation.

Interactive Learning-Based Realizability Interpretation for Heyting Arithmetic with $\mathtt{EM}_1$

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Abstract. We interpret classical proofs as constructive proofs (with constructive rules for $\vee, \exists$) over a suitable structure $\mathcal{N}$ for the language of natural numbers and maps of Gödel's system $\mathcal{T}$. We introduce a new Realization semantics we call "Interactive learning-based Realizability", for Heyting Arithmetic plus $\mathtt{EM}_1$ (Excluded middle axiom restricted to Σ^0_1 formulas). Individuals of $\mathcal{N}$ evolve with time, and realizers may "interact" with them, by influencing their evolution. We build our semantics over Avigad's fixed point result [1], but the same semantics may be defined over different constructive interpretations of classical arithmetic (in [7], continuations are used). Our notion of realizability extends Kleene's realizability and differs from it only in the atomic case: we interpret atomic realizers as "learning agents".

P.-L. Curien (Ed.): TLCA 2009, LNCS 5608, pp. 20–34, 2009.

JOURNAL ARTICLE

An analysis of the Podelski–Rybalchenko termination theorem via bar recursion

Stefano Berardi, Paulo Oliva, Silvia Steila

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Abstract

We present an effective proof (with explicit bounds) of the Podelski and Rybalchenko Termination Theorem. The sub-recursive bounds we obtain make use of bar recursion, in the form of the product of selection functions, as this is used to interpret the Weak Ramsey Theorem for pairs. The construction can be seen as calculating a modulus of well-foundedness for a given program given moduli of well-foundedness for the disjunctively well-founded finite set of covering relations. When the input moduli are in system T, this modulus is also definable in system T by a result of Schwichtenberg on bar recursion.

Issue Section: Special Issue Article

Plan

- Realizability Interpretations
  (background)
- Uniform Interpretations of Quantifiers
  (a bit of history...)
- A Uniform Realizability Interpretation
  (parametrised by a base interpretation)
- Some Base Interpretations
  (examples including the 'learning-based realizability')

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Realizability Interpretations

- Key idea: "Skolemize" A as $\exists \vec{x} A_{ef}(\vec{x})$

$$A^r \; :\equiv \{\vec{t} \mid A_{ef}(\vec{t})\}$$

Definition (Realizability Interpretation).

Define $\boldsymbol{a} \; \mathrm{r} \; A$ by induction on the formula A:

$$\begin{aligned}
\langle\,\rangle \; \mathrm{r} \; P(\vec{x}) \quad &:\equiv \quad P(\vec{x}) \\
\boldsymbol{a}, \boldsymbol{b} \; \mathrm{r} \; A \wedge B \quad &:\equiv \quad (\boldsymbol{a} \; \mathrm{r} \; A) \wedge (\boldsymbol{b} \; \mathrm{r} \; B) \\
\boldsymbol{f} \; \mathrm{r} \; A \to B \quad &:\equiv \quad \forall \boldsymbol{a}((\boldsymbol{a} \; \mathrm{r} \; A) \to (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge \, \boldsymbol{f} \bullet \boldsymbol{a} \; \mathrm{r} \; B)) \\
k, \boldsymbol{a} \; \mathrm{r} \; \exists n^{\mathbb{N}} A \quad &:\equiv \quad \boldsymbol{a} \; \mathrm{r} \; A[k/n] \\
\boldsymbol{f} \; \mathrm{r} \; \forall n^{\mathbb{N}} A \quad &:\equiv \quad \forall n^{\mathbb{N}}(\boldsymbol{f}(n) \; \mathrm{r} \; A)
\end{aligned}$$

- So, $A^r \equiv \{\boldsymbol{a} \mid \boldsymbol{a} \; \mathrm{r} \; A\}$, i.e. $A_{ef}(\vec{t})$ can be defined inductively

Realizability Interpretations

- Skolemization relies on **AC**

$$\forall x^\rho \exists y^\tau A(x,y) \to \exists f^{\rho\to\tau} \forall x^\rho A(x, f(x))$$

- In general we do not have

$$\forall x^\rho \exists y^\tau A(x,y) \to \exists y^\tau \forall x^\rho A(x,y)$$

- But, sometimes we do!
  - **Pointwise continuity** implies **uniform continuity**

    $\forall f \exists n \forall g \ldots \to \exists n \forall f, g \ldots$

  - **Bounded collection** (when $A(n,m)$ monotone in m)

    $\forall n \leq k \exists m\, A(n,m) \to \exists m \forall n \leq k\, A(n,m)$

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Uniform Interpretations of Quantifiers

Typed lambda-calculus in classical Zermelo-Frænkel set theory

2001

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In this paper, we develop a system of typed lambda-calculus for the Zermelo-Frænkel set theory, in the framework of classical logic. The first, and the simplest system of typed lambda-calculus is the *system of simple types*, which uses the intuitionistic propositional calculus, with the only connective $\to$. It is very important, because the well known Curry-

The definition is given by induction on F :

$$|F \to G| = (|F| \to |G|) \ ; \ |\forall x\, F| = \bigcap_a |F[a/x]|.$$

Therefore :

$t \Vdash (F \to G)$ is the formula $(\forall u \in \Lambda)(u \Vdash F \to tu \Vdash G)$;
$t \Vdash \forall x\, F$ is the formula $\forall x(t \Vdash F)$.

Uniform Interpretations of Quantifiers

2003

UNIFORM HEYTING ARITHMETIC

ULRICH BERGER

Dedicated to Helmut Schwichtenberg on his 60th Birthday

Abstract. We present an extension of Heyting Arithmetic in finite types called *Uniform Heyting Arithmetic* (HA^{u}) that allows for the extraction of optimized programs from constructive and classical proofs. The system HA^{u} has two sorts of first-order quantifiers: ordinary quantifiers governed by the usual rules, and uniform quantifiers subject to stronger variable conditions expressing roughly that the quantified object is not computationally used in the proof. We combine a Kripke-style Friedman/Dragalin translation which is inspired by work of Coquand and Hofmann and a variant of the refined A-translation due to Buchholz, Schwichtenberg and the author to extract programs from a rather large class of classical first-order proofs while keeping explicit control over the levels of recursion and the decision procedures for predicates used in the extracted program.

$$r\ \mathbf{mr}\ \exists x^{\rho}\, A = \begin{cases} \mathsf{p}_1(r)\ \mathbf{mr}\ A[\mathsf{p}_0(r)/x] & \text{if } A \text{ is non-Harrop} \\ \epsilon\ \mathbf{mr}\ A[r/x] & \text{if } A \text{ is Harrop} \end{cases}$$

$$r\ \mathbf{mr}\ \mathsf{Q}A = \mathsf{Q}(r\ \mathbf{mr}\ A) \text{ where } \mathsf{Q} \in \{\{\forall x\}, \{\exists x\}\}$$

Uniform Interpretations of Quantifiers

A functional interpretation for nonstandard arithmetic

2012

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A B S T R A C T

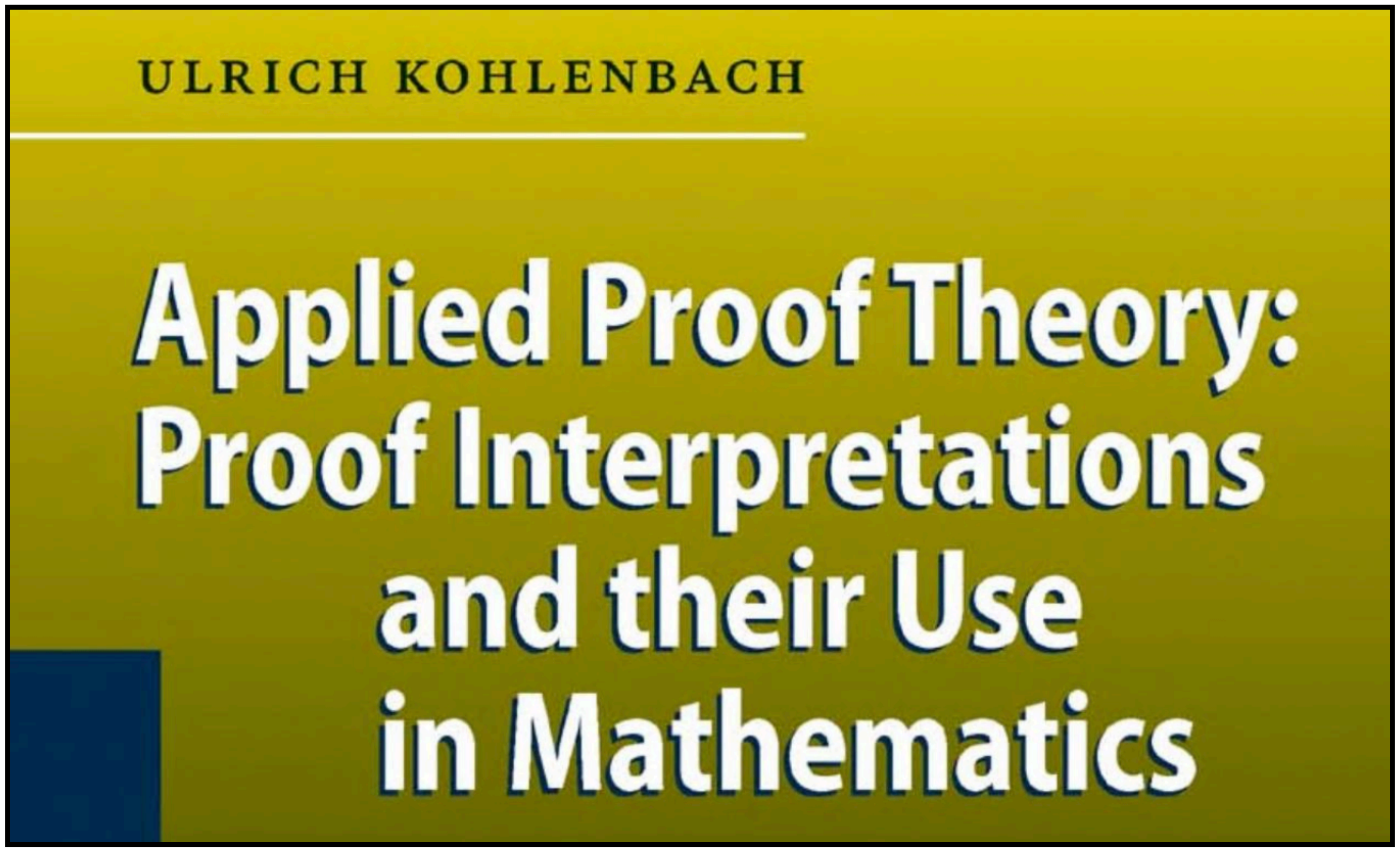
We introduce constructive and classical systems for nonstandard arithmetic and show how variants of the functional interpretations due to Gödel and Shoenfield can be used to rewrite proofs performed in these systems into standard ones. These functional interpretations show in particular that our nonstandard systems are conservative extensions of E-HA$^\omega$ and E-PA$^\omega$, strengthening earlier results by Moerdijk and Palmgren, and Avigad and Helzner. We will also indicate how our rewriting algorithm can be used for term extraction purposes. To conclude the paper, we will point out some open problems and directions for future research, including some initial results on saturation principles.

$$\begin{aligned}
\underline{s}\ \mathrm{hr}\, \exists x\, \Phi(x) &:\equiv \exists x\, \big(\underline{s}\, \mathrm{hr}\, \Phi(x)\big), \\
\underline{s}\ \mathrm{hr}\, \forall x\, \Phi(x) &:\equiv \forall x\, \big(\underline{s}\, \mathrm{hr}\, \Phi(x)\big), \\
s, \underline{t}\ \mathrm{hr}\, \exists^{\mathrm{st}} x\, \Phi(x) &:\equiv \exists s' \in s\, \big(\underline{t}\, \mathrm{hr}\, \Phi(s')\big), \\
\underline{s}\ \mathrm{hr}\, \forall^{\mathrm{st}} x\, \Phi(x) &:\equiv \forall^{\mathrm{st}} x\, \big(\underline{s}[x]\, \mathrm{hr}\, \Phi(x)\big).
\end{aligned}$$

(internal)

(external)

Uniformity in Proof Mining



This book gives an introduction to so-called proof interpretations, more specifically various forms of realizability and functional interpretations, and their use in mathematics. Whereas earlier treatments of these techniques (e.g. [362, 264, 121, 365, 7]) emphasize foundational and logical issues the focus of this book is on applications of the methods to extract new effective information such as computable uniform bounds from given (typically ineffective) proofs. This line of research, which has its roots in G. Kreisel's pioneering work on 'unwinding of proofs' from the 50's, has

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Heyting Arithmetic

Definition (Heyting arithmetic).

Assume $\mathbf{HA}$ formalised with three predicate symbols

- Falsity $\bot$ — nullary
- Natural number $\mathbb{N}(n)$ — unary
- Equality $n = m$ — binary

Notation.

$\forall n^{\mathbb{N}} A(n)$ is an abbreviation for $\forall n(\mathbb{N}(n) \to A(n))$

$\exists n^{\mathbb{N}} A(n)$ is an abbreviation for $\exists n(\mathbb{N}(n) \wedge A(n))$

A Uniform Realizability Interpretation

Definition (Base Interpretation).

Associate to each n-ary predicate symbol P an $(n+m)$-ary relation

$$\vec{x} \lhd_P \boldsymbol{a}$$

between individuals and P-realizers (or P-bounds).

Examples.

For the unary predicate $\mathbb{N}(n)$ we could take $n \lhd_{\mathbb{N}} \cdot$ to be:

$$
\begin{aligned}
n \lhd_{\mathbb{N}} m &:\equiv n = m && \text{(precise)} \\
n \lhd_{\mathbb{N}} m &:\equiv n \leq m && \text{(bounded)} \\
n \lhd_{\mathbb{N}} S &:\equiv n \in S && \text{(Herbrand)} \\
n \lhd_{\mathbb{N}} \langle\,\rangle &:\equiv \text{true} && \text{(uniform)}
\end{aligned}
$$

A Uniform Realizability Interpretation

Definition (Uniform Realizability Interpretation).

Given a base interpretation. Let:

$$
\begin{aligned}
\boldsymbol{a} \text{ ur } P(\vec{x}) &:\equiv \vec{x} \lhd_P \boldsymbol{a} \\
\boldsymbol{a}, \boldsymbol{b} \text{ ur } A \wedge B &:\equiv (\boldsymbol{a} \text{ ur } A) \wedge (\boldsymbol{b} \text{ ur } B) \\
\boldsymbol{f} \text{ ur } A \to B &:\equiv \forall \boldsymbol{a}((\boldsymbol{a} \text{ ur } A) \to (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge \boldsymbol{f} \bullet \boldsymbol{a} \text{ ur } B)) \\
\boldsymbol{a} \text{ ur } \exists x A &:\equiv \exists x (\boldsymbol{a} \text{ ur } A) \\
\boldsymbol{a} \text{ ur } \forall x A &:\equiv \forall x (\boldsymbol{a} \text{ ur } A)
\end{aligned}
$$

It follows that...

$$
\begin{aligned}
\boldsymbol{a}, \boldsymbol{b} \text{ ur } \exists n^{\mathbb{N}} A &:\equiv \exists n \lhd_{\mathbb{N}} \boldsymbol{a} \, (\boldsymbol{b} \text{ ur } A) \\
\boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &:\equiv \forall \boldsymbol{a} \forall n \lhd_{\mathbb{N}} \boldsymbol{a} \, (\boldsymbol{f} \bullet \boldsymbol{a} \downarrow \wedge \boldsymbol{f} \bullet \boldsymbol{a} \text{ ur } A)
\end{aligned}
$$

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Kreisel Modified Realizability

Definition (Kreisel Base Interpretation).

Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\bot} \langle\,\rangle \quad &:\equiv \quad \bot \\ n \lhd_{\mathbb{N}} m^{\mathbb{N}} \quad &:\equiv \quad n = m \\ (n, m) \lhd_{=} \langle\,\rangle \quad &:\equiv \quad n = m \end{aligned}$$

It follows that...

$$\begin{aligned} n^{\mathbb{N}}, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) \quad &\Leftrightarrow \quad \boldsymbol{a} \text{ ur } A(n) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A \quad &\Leftrightarrow \quad \forall n^{\mathbb{N}}(\boldsymbol{f}(n) \text{ ur } A) \end{aligned}$$

Herbrand Realizability

Definition (Herbrand Base Interpretation).

Assume an extra unary predicate $\mathrm{std}(n)$ (for n is a **standard number**). Let:

$$\begin{aligned} \langle\,\rangle \lhd_{\bot} \langle\,\rangle &:\equiv \bot \\ n \lhd_{\mathbb{N}} \langle\,\rangle &:\equiv \mathrm{true} \\ n \lhd_{\mathrm{std}} S &:\equiv n \in S \\ (n, m) \lhd_{=} \langle\,\rangle &:\equiv n = m \end{aligned}$$

It follows that...

$$\begin{aligned} \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \exists n^{\mathbb{N}}(\boldsymbol{a} \text{ ur } A(n)) \\ \boldsymbol{a} \text{ ur } \forall n^{\mathbb{N}} A(n) &\Leftrightarrow \forall n^{\mathbb{N}}(\boldsymbol{a} \text{ ur } A(n)) \\ S^{\mathbb{N}^*}, \boldsymbol{a} \text{ ur } \exists n^{\mathrm{std}} A(n) &\Leftrightarrow \exists n \in S\, (\boldsymbol{a} \text{ ur } A(n)) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathrm{std}} A &\Leftrightarrow \forall S\, \forall n \in S\, (\boldsymbol{f}(S) \text{ ur } A) \end{aligned}$$

Aschieri-Berardi Learning Realizability

Definition (Aschieri-Berardi Base Interpretation).

Assume a set of states $\mathbf{S}$. Parametrised by an $s \in \mathbf{S}$, let:

$$\begin{aligned} \langle\,\rangle \lhd_{\perp} \gamma^{\mathbf{S}\to\mathbf{S}} &:\equiv \gamma(s) \neq s \\ n \lhd_{\mathbb{N}} \alpha^{\mathbf{S}\to\mathbb{N}} &:\equiv n = \alpha(s) \\ (n, m) \lhd_{=} \gamma^{\mathbf{S}\to\mathbf{S}} &:\equiv \gamma(s) = s \to n = m \end{aligned}$$

It follows that...

$$\begin{aligned} \alpha^{\mathbf{S}\to\mathbb{N}}, \boldsymbol{a} \text{ ur } \exists n^{\mathbb{N}} A(n) &\Leftrightarrow \boldsymbol{a} \text{ ur } A(\alpha(s)) \\ \boldsymbol{f} \text{ ur } \forall n^{\mathbb{N}} A &\Leftrightarrow \forall n^{\mathbb{N}}(\boldsymbol{f}(n) \text{ ur } A) \end{aligned}$$

Summary

- Quantifiers are "naturally" **uniform** (non-computational)
- Qualified quantifications (e.g. $\exists n^{\mathbb{N}} A(n)$) carry computational content because of the qualifying predicate $\mathbb{N}(n)$
- Recent work with Fernando Ferreira on uniform interpretations:
  - New interpretations of function spaces $\rho \to \tau$
  - Functional interpretation of extensionality
  - Systematic treatment of bounded (uniform) quantifiers